INFINITE NUMERICAL SEQUENCES OF EXPONENTS IN THE PRIME FACTORIZATION OF INFINITE NUMERICAL SEQUENCES OF FACTORIALS OF THE GIVEN PRIME NUMBER p.
THE NUMERICAL SEQUENCES $\left(A_{p,(r_n)}\right)$ AND $\left(B_{p,(r_n)}\right)$ FOR THE GIVEN PRIME NUMBER p FOR A CERTAIN NUMERICAL SEQUENCE (r_n)
VERSION I.

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Abstract. In the prime factorization of $n!$ one could determine **one** corresponding exponent $ord_p n!$ of the given prime number p for **one** certain positive integer $0 \leq n \leq \infty$ using Chebyshev's formula or the derivation of his formula. In contrast to that one can now find in prime factorization the corresponding numerical sequence of exponents $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right)$ of **every** single factorial of the given numerical sequence of factorials $\left(r_{n!}\right) = \left(0!, 1!, 2!, ..., n!\right)$ using the numerical sequences $\left(A_{p,(r_n)}\right)$ and $\left(B_{p,(r_n)}\right)$ of the given prime number p for the certain numerical sequence $(r_n) = (0, 1, 2, ..., n)$ of the members $0 \leq n \leq \infty$. (This means that instead of $ord_p n!$ one receives $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right)$).

In the Appendix there are programmes verifying the results of the numerical sequences $\left(A_{p,(r_n)}\right)$, (Variation II.c) ('A.py') and $\left(B_{p,(r_n)}\right)$ (Variation II.c) ('B.py'). Also a programme is given to compare these results to those of Chebyshev's Formula (1.2) ('Chebyshev.py'). The results for all formulae are verified up to $n = 2^{20}$ for all prime numbers from $p_1 = 2$ up to $p \leq n$.

1. Introduction

The numerical sequences of the given prime number p from a certain numerical sequence (r_n) are denoted as $\left(A_{p,(r_n)}\right)$ and $\left(B_{p,(r_n)}\right)$, where index p represents the numerical sequence for the given prime number p and index (r_n) the numerical sequence $(r_n) = (0, 1, 2, ..., n)$ of the positive integers $0 \leq n \leq \infty$. In the prime factorization of $n!$ (1.1) one could determine **one** corresponding exponent $ord_p n!$ of the given prime number p for **one** certain positive integer $0 \leq n \leq \infty$, using Chebyshev's formula (1.2) or the derivation of his formula (2.2). In contrast to that one now finds in prime factorization the corresponding numerical sequence of exponents $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right)$ of **every** single factorial of the given numerical sequence of factorials $\left(r_{n!}\right) = \left(0!, 1!, 2!, ..., n!\right)$, using the numerical sequences $\left(A_{p,(r_n)}\right)$ and $\left(B_{p,(r_n)}\right)$ of a given prime number p for the certain numerical sequence $(r_n) = (0, 1, 2, ..., n)$ of the members consisting of the positive integers $0 \leq n \leq \infty$. One applies here the formulae (2.12), (2.13), (2.19), (2.20), (2.21) for

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the numerical sequence $\left(A_{p,(r_n)}\right)$, (3.9), (3.10), (3.11), (3.12), (3.21), (3.22), (3.23), (3.24), (3.25) for the numerical sequence $\left(A_{p,\left(r_{n_2}\right)}\right)$ and (4.2), (4.7), (4.8) for the numerical sequence $\left(B_{p,(r_n)}\right)$, (5.1), (5.6), (5.7), (5.8) for the numerical sequence $\left(B_{p,(r'_n)}\right)$.

Note 1. *If one determines in the prime factorization of* $n!$ ***every*** *exponent* $ord_p n!$ *of the given prime number* p *for every positive integer* $n = 0, 1, 2, ..., (p^{\eta} - 1)$, *with* $\eta = 0, 1, 2, ..., \infty$, *using Chebyshevs formula or its derivation, one obtains a numerical sequence of exponents* $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., (p^{\eta} - 1)!\right)$ *for a numerical sequence of factorials* $\left(r_{n!}\right) = \left(0!, 1!, 2!, ..., (p^{\eta} - 1)!\right)$ *for the numerical sequence* $(r_n) = (0, 1, 2, ..., (p^{\eta} - 1))$. *This possibility is used in order to double-check the results obtained from using the formula in this paper.*

In Section 2 it is shown how to find an infinite numerical sequence of exponents $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., (p^{\eta} - 1)!\right)$ for an infinite numerical sequence of factorials $\left(r_{n!}\right) = \left(0!, 1!, 2!, ..., (p^{\eta} - 1)!\right)$ for an infinite numerical sequence $(r_n) = (0, 1, 2, ..., (p^{\eta} - 1))$, with $\eta = 0, 1, 2, ..., \infty$. For that one uses the numerical sequences $(a) = (0, 1, 2, ..., (p - 1))$. The numerical sequence $\left(A_{p,(r_n)}\right)$ of the given prime number p from a certain numerical sequence (r_n) one obtains as a sum of the numerical sequences $(a) = (0, 1, 2, ..., (p - 1))$:

Variation I. (cf. Formulae (2.12) and (2.13))

$$(\overset{\triangleq p^{\eta}}{A}_{p,(r_n)}) = (\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{\eta}{(a)}),$$

then

$$ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., (p^{\eta} - 1)!\right) = \frac{1}{p-1} \cdot \left((r_n) - \left(A_{p,(r_n)}\right)\right).$$

Apart from that, in the same section Variation II.c is developed. It clarifies that it is more convenient to use the following development of the numerical sequence $\left(A_{p,\left(r_{n_2}\right)}\right)$ of the given prime number p from a certain numerical sequence $\left(r_{n_2}\right) = \left(0, 1, 2, ..., \left(p^{p^{\eta+1}} - 1\right)\right)$ step by step (cf. Formulae (2.19), (2.20) and (2.21)):

$$((\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + (\underset{2,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + (\underset{3,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + \underset{...,}{\overset{...}{...}} + (\underset{p}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)})) = (\overset{\triangleq p^{p^{\eta+1}}}{A}_{p,\left(r_{n_2}\right)}),$$

where

$$(\overset{\triangleq p^{p^{\eta}}}{A}_{p,\left(r_{n_1}\right)}) = (\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}),$$

then

$$ord_p\left(r_{(n_2)!}\right) = ord_p\left(0!, 1!, 2!, ..., \left(p^{p^{\eta+1}} - 1\right)!\right) = \frac{1}{p-1} \cdot \left(\left(r_{n_2}\right) - \left(A_{p,\left(r_{n_2}\right)}\right)\right).$$

If $0 \leq n_1 \leq \left(p^{p^{\eta}} - 1\right)$, $0 \leq n_2 \leq \left(p^{p^{\eta+1}} - 1\right)$, $(a) = (0, 1, 2, ..., (p - 1))$, $\left(r_{n_1}\right) = \left(0, 1, 2, ..., \left(p^{p^{\eta}} - 1\right)\right)$, $\alpha_{p,\left(p^{p^{\eta}}-1\right)} = p^{\eta} \cdot (p - 1)$, $Nr.\left(\alpha_{p,\left(p^{p^{\eta}}-1\right)}\right) = p^{p^{\eta}} - 1$, $\left(r_{n_2}\right) = \left(0, 1, 2, ..., \left(p^{p^{\eta+1}} - 1\right)\right)$, $\alpha_{p,\left(p^{p^{\eta+1}}-1\right)} = p^{\eta+1} \cdot (p - 1)$, $0 \leq \eta \leq \infty$, $p \geq 2$,

$Nr.\left(\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}\right) = p^{p^{\eta+1}} - 1,\ \alpha_{p,p^{\eta}} = 1,\ Nr.\left(\alpha_{p,0}\right) = 0,\ \alpha_{p,0} = 0.$

In the third section the author describes how to find an infinite numerical sequence of exponents $ord_p\left(r_{(pn)!}\right) = ord_p\left(p \cdot (0,1,2,...,(p^{\eta}-1))\right)!$ for an infinite numerical sequence of factorials $\left(r_{(pn)!}\right) = \left(p \cdot (0,1,2,...,(p^{\eta}-1))\right)!$ for an infinite numerical sequence $(r_{pn}) = p \cdot (0,1,2,...,(p^{\eta}-1))$, with $\eta = 0,1,2,...,\infty$.

If using the numerical sequence $\left(A_{p,(r_{pn})}\right)$ instead of the numerical sequence $\left(A_{p,(r_n)}\right)$, where $(r_n) = (0,1,2,...,(p^{\eta}-1))$ and where $(r_{pn}) = p \cdot (0,1,2,...,(p^{\eta}-1))$, one can shorten the calculation p-times. The exponent obtained this way in the numerical sequence of exponents is written p-times repeated. Then one obtains the numerical sequence $(r'_n) = \left(0,1,2,...,\left(p^{\eta+1}+p-1\right)\right)$ instead of the numerical sequence $(r_{pn}) = p \cdot (0,1,2,...,(p^{\eta}-1))$. The numerical sequence $\left(A_{p,(r_{pn})}\right)$ one has to consider as sum of numerical sequences $(a) = (0,1,2,...,(p-1))$:

Variation I. (cf. Formulae (3.9), (3.10), (3.11) and (3.12))

$$(\overset{\triangleq p^{\eta}}{A}_{p,(r_n)}) = ((\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{\eta}{(a)}),$$

$$(\overset{\triangleq p^{\eta}}{A}_{p,(r_{pn})}) = ((\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{\eta}{(a)}),$$

where

$$\begin{aligned} 1)\ ord_p\left(r_{(pn)!}\right) &= ord_p\left(p \cdot (0,1,2,...,(p^{\eta}-1))\right)! = \\ &= \frac{1}{p-1} \cdot \left(\left(r_{(pn)}\right) - \left(A_{p,\left(r_{(pn)}\right)}\right)\right). \end{aligned}$$

If one writes every single exponent $ord_p\left(r_{(p \cdot n)!}\right)$, one has obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $(r'_{n!}) = \left(0!,1!,2!,...,\left(p^{\eta+1}-1\right)!\right)$, i.e.:

$$2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!,1!,2!,...,\left(p^{\eta+1}-1\right)!\right)$$

under the condition that $(r_{pn}) = p \cdot (0,1,2,...,(p^{\eta}-1))$, $(r'_n) = \left(0,1,2,...,\left(p^{\eta+1}-1\right)\right)$, $p \geq 2$, $0 \leq \eta \leq \infty$.

Again in the same section a Variation II.c is developed.

Variation II.c (cf. Formulae (3.21), (3.22), (3.23), (3.24) and (3.25))

If

$$((\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{p\cdot n_1})}) + (\underset{2,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{p\cdot n_1})}) + (\underset{3,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{p\cdot n_1})}) + \underset{...,}{\overset{...}{...}} + (\underset{p}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{p\cdot n_1})})) = (\overset{\triangleq p^{p^{\eta+1}}}{A}_{p,(r_{pn_2})}),$$

$$(\overset{\triangleq p^{\eta}}{A}_{p,(r_{n_1})}) = ((\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}),$$

and

$$(\overset{\triangleq p^{\eta}}{A}_{p,(r_{pn_1})}) = ((\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}),$$

then

$$
\begin{aligned}
1)\ \ ord_p\left(r_{(p\cdot n_2)!}\right) &= ord_p\left((0\cdot p)!,(1\cdot p)!,(2\cdot p)!,...,\left(\left(p^{p^{\eta+1}}-1\right)\cdot p\right)!\right) = \\
&= \frac{1}{p-1}\cdot\left(p\cdot\left(r_{n_2}\right)-\left(A_{p,\left(r_{(p\cdot n_2)}\right)}\right)\right),
\end{aligned}
$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n_2)!}\right)$, we have obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $\left(r'_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}+1}-1\right)\right)$, i.e. if considering Remark 2 one obtains:

$$
2)\ \ ord_p\left(r'_{(n_2)!}\right)=\left(0!,1!,2!,...,\left(p^{p^{\eta+1}+1}-1\right)!\right).
$$

Additionally to that one considers Remark 1, under the condition that

$$
\begin{aligned}
&n_1=0,1,2,...,\left(p^{p^{\eta}}-1\right),n_2=0,1,2,...,\left(p^{p^{\eta+1}}-1\right),\\
&Nr.\left(\alpha_{p,p\cdot\left(p^{p^{\eta}}-1\right)}\right)=p\cdot\left(p^{p^{\eta}}-1\right),\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}=p^{\eta}\cdot(p-1),\\
&Nr.\left(\alpha_{p,p\cdot\left(p^{p^{\eta+1}}-1\right)}\right)=p\cdot\left(p^{p^{\eta+1}}-1\right),\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}=p^{\eta+1}\cdot(p-1),\\
&0\leq\eta\leq\infty,p\geq2,\alpha_{p,p^{\eta}}=1,Nr.\left(\alpha_{p,0}\right)=0,\alpha_{p,0}=0,\\
&\left(r_{n_1}\right)=\left(0,1,2,...,\left(p^{p^{\eta}}-1\right)\right),\\
&\left(r_{(p\cdot n_1)}\right)=\left((p\cdot0),(p\cdot1),(p\cdot2),...,\left(p\cdot\left(p^{p^{\eta}}-1\right)\right)\right),\\
&\left(r_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right),\\
&\left(r_{(p\cdot n_2)}\right)=\left((p\cdot0),(p\cdot1),(p\cdot2),...,\left(p\cdot\left(p^{p^{\eta+1}}-1\right)\right)\right),\\
&\left(r_{(p\cdot n_2)!}\right)=\left((0\cdot p)!,(1\cdot p)!,(2\cdot p)!,...,\left(\left(p^{p^{\eta+1}}-1\right)\cdot p\right)!\right),\\
&\left(r'_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}+1}-1\right)\right),\\
&\left(r'_{(n_2)!}\right)=\left(0!,1!,2!,...,\left(p^{p^{\eta+1}+1}-1\right)!\right),\\
&(a)=(0,1,2,...,(p-1)).
\end{aligned}
$$

In the fourth section another kind of numerical sequences is used, namely $\left(B_{p,(r_n)}\right)$. The exponent $ord_p n!$ for a certain number n in the prime factorization of $n!$ is the sum of all previous members of the numerical sequence $\left(B_{p,(r_n)}\right)$ and the last member $\beta_{p,p^{\eta}}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$, which corresponds to the given number n. Every member $\beta_{p,(r_n)}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ has got a certain number. We will determine the number of every member $\beta_{p,(r_n)}$, in order to insert every certain member $\beta_{p,(r_n)}$ at a set position in the numerical sequence $\left(B_{p,(r_n)}\right)$. This method is used to find all exponents $ord_p\left(r_{n!}\right)=ord_p\left(0!,1!,2!,...,(p^{\eta})!\right)$ for the given prime number p in the prime factorization of the given numerical sequence of factorials $\left(r_{n!}\right)=\left(0!,1!,2!,...,(p^{\eta})!\right)$. The numerical sequence $\left(B_{p,(r_n)}\right)$ is (cf. Formulae (4.2), (4.7) und (4.8)):

$$\left(B_{p,(r_n)}\right) = \left(\beta_{p,(r_n)}\right) = \left(\left(\beta_{p,0}\right), \beta_{p,1}, \beta_{p,2}, ..., \beta_{p,p^\eta}\right) =$$
$$= (\overset{0,}{(0)}, \overset{1,}{0}, \overset{...,}{...}, \overset{(p-1),}{0}, \overset{p,}{1}, \overset{(p+1),}{0}, \overset{...,}{...}, \overset{(2p-1),}{0}, \overset{2p,}{1}, \overset{(2p+1),}{0}, \overset{...,}{...}, \overset{\left(p^2-1\right),}{0}, \overset{p^2,}{2}, \overset{\left(p^2+1\right),}{0}, \overset{...,}{...}, \overset{(p^\eta-1),}{0}, \overset{p^\eta,}{\eta}).$$

The numbers above each member are defined as follows:

a) The number of the first member $\beta_{p,0}$ is:

$$Nr.\left(\beta_{p,0}\right) = Nr.\,(0) = (0) \text{ Definition,}$$
$$\left(\beta_{p,0}\right) = (0) \text{ Definition,}$$

b) the numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1}, \beta_{p,2}, \beta_{p,3}, ..., \beta_{p,(p^\eta-1)}\right)$ is:

$$(Nr.\,(i)) = p^i \cdot \left[(1, 2, ..., (p-1)) + \left(p \cdot \left(0, 1, 2, ..., \left(p^{\eta-i-1} - 1\right)\right)\right)\right].$$

At the same time each member is a number i , with $i = 0, 1, 2, ..., (\eta - 1)$

c) The number of the last member $\beta_{p,(p^\eta)}$ is:

$$Nr.\,(\eta) = p^\eta.$$

The last member $\beta_{p,(p^\eta)}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is simultaneously the last number η.

Then one obtains the numerical sequence of exponents

$$\begin{aligned} ord_p\left(r_{n!}\right) &= ord_p\left(0!, 1!, 2!, ..., (p^\eta)!\right) = \\ &= \left(\sum_{x=0}^{0} \beta_{p,x}, \sum_{x=0}^{1} \beta_{p,x}, \sum_{x=0}^{2} \beta_{p,x}, ..., \sum_{x=0}^{n=p^\eta} \beta_{p,x}\right) = \\ &= \left(\left(\beta_{P,0}\right), \left(\beta_{p,0} + \beta_{p,1}\right), \left(\beta_{p,0} + \beta_{p,1} + \beta_{p,2}\right), ... \right. \\ &\quad \left. ..., \left(\beta_{p,0} + \beta_{p,1} + \beta_{p,2} + ... + \beta_{p,p^\eta}\right)\right). \end{aligned}$$

Additionally, one considers Remark 1 on page 20, under the condition that $0 \leq n \leq p^\eta$, $0 \leq i \leq (\eta - 1)$, $0 \leq \eta \leq \infty$, $(r_n) = (0, 1, 2, ..., p^\eta)$, $(r_{n!}) = (0!, 1!, 2!, ..., (p^\eta)!)$, $\beta_{p,1} = 0$, $\beta_{p,p} = 1$, $\beta_{p,p^2} = 2$,...,$\beta_{p,p^\eta} = \eta$, $p \geq 2$, $Nr.\left(\beta_{p,0}\right) = (0)$ Definition and $\left(\beta_{p,0}\right) = (0)$ Definition.

Section 5 deals with the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$. One uses this numerical sequence to obtain a better result with less effort. It would be very unpractical to have an addition of zeros. For this reason one can spare oneself the work by repeating every exponent p-times for the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ of the exponents $ord_p\left(r_{(pn)!}\right) = ord_p\left(0!, p!, (2p)!, ..., \left(p^{\eta+1}\right)!\right)$. Thus, for the numerical sequence $\left(B_{p,\left(r'_{n!}\right)}\right)$ one obtains the numerical sequence of exponents $ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., \left(p^{\eta+1} + p - 1\right)!\right)$.

The numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is composed as follows (cf. Formulae (5.1), (5.6), (5.7) and (5.8)):

$$\begin{aligned}\left(B_{p,\left(r_{(pn)}\right)}\right) &= \left(\beta_{p,\left(r_{(pn)}\right)}\right) = \left(\left(\beta_{p,(p\cdot 0)}\right), \beta_{p,(p\cdot 1)}, \beta_{p,(p\cdot 2)}, ..., \beta_{p,(p^{\eta}\cdot p)}\right) = \\ &= \left(\overset{0,}{(0),} \overset{p,}{1,} \overset{...,}{...,} \overset{(p^2-p),}{1,} \overset{p^2,}{2,} \overset{(p^2+p),}{1,} \overset{...,}{...,} \overset{(p^{\eta+1}-p),}{1,} \overset{p^{\eta+1};}{(\eta+1)}\right).\end{aligned}$$

The numbers above each member are defined as follows:

a) The number of the first member $\beta_{p,0}$ is:

$$\begin{aligned}&Nr.\left(\beta_{P,0}\right) = Nr.\,(0) = (0)\,\text{Definition},\\ &\left(\beta_{P,0}\right) = (0)\,\text{Definition},\end{aligned}$$

b) the numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1\cdot p}, \beta_{p,2\cdot p}, \beta_{p,3\cdot p}, ..., \beta_{p,(p^{\eta}-1)\cdot p}\right)$ is:

$$(Nr.\,(i+1)) = p^{i+1}\cdot\left[(1,2,...,(p-1)) + \left(p\cdot\left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right].$$

At the same time each member is a number i, with $i = 0,1,2,...,(\eta-1)$.

c) The number of the last member $\beta_{p,p^{\eta+1}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is:

$$Nr.\,(\eta+1) = p^{\eta+1}.$$

The last member $\beta_{p,(p^{\eta+1})}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is simultaneously the last number $\eta+1$.

Then one obtains the numerical sequence of exponents

$$\begin{aligned}1)\ ord_p\left(r_{(pn)!}\right) &= ord_P\left((p\cdot 0)!, (p\cdot 1)!, (p\cdot 2)!, ..., \left(p^{\eta+1}\right)!\right) = \\ &= \left(\sum_{x=0}^{(p\cdot 0)}\beta_{p,(px)}, \sum_{x=0}^{(p\cdot 1)}\beta_{p,(px)}, \sum_{x=0}^{(p\cdot 2)}\beta_{p,(px)}, ..., \sum_{x=0}^{(p\cdot n)=p^{\eta+1}}\beta_{p,(px)}\right) = \\ &= \left(\left(\beta_{p,(p\cdot 0)}\right), \left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}\right), \left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}+\beta_{p,(p\cdot 2)}\right), ...\right.\\ &\quad \left. ..., \left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}+\beta_{p,(p\cdot 2)}+...+\beta_{p,(p^{\eta+1})}\right)\right).\end{aligned}$$

If writing every exponents p-times repeated in the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ of the exponents $ord_p\left(r_{(pn)!}\right) = ord_p\left(0!, p!, (2p)!, ..., \left(p^{\eta+1}\right)!\right)$, one obtains for the numerical sequence $\left(B_{p,\left(r'_{n!}\right)}\right)$ the numerical sequence of exponents:

$$2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., \left(p^{\eta+1}+p-1\right)!\right).$$

Additionally, one considers Remark 1 on page 20, under the condition that

$n = 0,1,2,...,p^{\eta}, i = 0,1,2,...,(\eta-1), \eta = 0,1,2,...,\infty, (r_n) = (0,1,2,...,p^{\eta})$,
$(r'_n) = \left(0,1,2,...,\left(p^{\eta+1}+p-1\right)\right), (r_{n!}) = (0!,1!,2!,...,(p^{\eta})!)$,
$(r'_{n!}) = \left(0!,1!,2!,...,\left(p^{\eta+1}+p-1\right)!\right), \left(r_{(p\cdot n)}\right) = \left(0,p,(2p),...,\left(p^{\eta+1}\right)\right)$,
$\left(r_{(p\cdot n)!}\right) = \left(0!,p!,(2p)!,...,\left(p^{\eta+1}\right)!\right)$,
$\beta_{p,p} = 1, \beta_{p,p^2} = 2, \beta_{p,p^3} = 3, ..., \beta_{p,p^{\eta+1}} = \eta+1, p \geq 2$,
$Nr.\left(\beta_{p,0}\right) = (0)$ Definition and $\left(\beta_{p,0}\right) = (0)$ Definition.

Here $ord_p n! = k$ is in the prime factorization of $n!$ for an arbitrary certain natural number n, if $n \in \mathbb{N}, \mathbb{N} = \{0, 1, 2, ..., i\}, i = 0, 1, 2, ..., \infty$, the maximum exponent for the given prime number p and therefore $n!$ is divisible by p^k. This is shown in Chebyshev's Formula (1.1), (1.2):

If

$$n! = \prod_{p=2}^{p \le n} p^{\sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor}, \tag{1.1}$$

then

$$ord_p n! = \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor . \tag{1.2}$$

To explain more clearly what the numerical sequences $\left(A_{p,(r_n)}\right)$ and $\left(B_{p,(r_n)}\right)$ of the given prime number p for a certain numerical sequence (r_n) are one has to look at Table 1 (T.1).

2. THE NUMERICAL SEQUENCES $\left(A_{p,(r_n)}\right)$ FOR THE GIVEN PRIME NUMBER p OF A CERTAIN NUMERICAL SEQUENCE (r_n).

In Table 1 (T.1) one first has to look at the numerical sequences $\left(A_{p,(r_n)}\right)$, where is shown how in the derivation of Chebyshev's formula the sums of the coefficients a_x are used to find in the prime factorization of $n!$ the corresponding exponent for the given prime number p for a certain positive integer $0 \le n \le \infty$.

Table 1 (T.1)

$P = 2$							
				A_2 ↓		**$\mathrm{ord}_2 n!$ ↓**	**B_2 ↓**
$0!$	⇒	$0 \cdot 2^0$	⇒	$(0 - \mathbf{0})$	=	**0**	> **0**
$1!$	⇒	$1 \cdot 2^0$	⇒	$(1 - \mathbf{1})$	=	**0**	> **0**
$2!$	⇒	$1 \cdot 2^1$	⇒	$(2 - \mathbf{1})$	=	**1**	> **1**
$3!$	⇒	$1 \cdot 2^1 + 1 \cdot 2^0$	⇒	$(3 - \mathbf{2})$	=	**1**	> **0**
$4!$	⇒	$1 \cdot 2^2$	⇒	$(4 - \mathbf{1})$	=	**3**	> **2**
$5!$	⇒	$1 \cdot 2^2 + 1 \cdot 2^0$	⇒	$(5 - \mathbf{2})$	=	**3**	> **0**
$6!$	⇒	$1 \cdot 2^2 + 1 \cdot 2^1$	⇒	$(6 - \mathbf{2})$	=	**4**	> **1**
$7!$	⇒	$1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$	⇒	$(7 - \mathbf{3})$	=	**4**	> **0**
$8!$	⇒	$1 \cdot 2^3$	⇒	$(8 - \mathbf{1})$	=	**7**	> **3**
$9!$	⇒	$1 \cdot 2^3 + 1 \cdot 2^0$	⇒	$(9 - \mathbf{2})$	=	**7**	> **0**
$10!$	⇒	$1 \cdot 2^3 + 1 \cdot 2^1$	⇒	$(10 - \mathbf{2})$	=	**8**	> **1**
$11!$	⇒	$1 \cdot 2^3 + 1 \cdot 2^1 + 1 \cdot 2^0$	⇒	$(11 - \mathbf{3})$	=	**8**	> **0**

				$A_2\downarrow$		$\mathbf{ord_2}\boldsymbol{n!}\downarrow$	$B_2\downarrow$
$12!$	$\Rightarrow$	$1\cdot 2^3+1\cdot 2^2$	$\Rightarrow$	$(12-\mathbf{2})$	$=$	$\mathbf{10}$	$>\ \mathbf{2}$
$13!$	$\Rightarrow$	$1\cdot 2^3+1\cdot 2^2+1\cdot 2^0$	$\Rightarrow$	$(13-\mathbf{3})$	$=$	$\mathbf{10}$	$>\ \mathbf{0}$
$14!$	$\Rightarrow$	$1\cdot 2^3+1\cdot 2^2+1\cdot 2^1$	$\Rightarrow$	$(14-\mathbf{3})$	$=$	$\mathbf{11}$	$>\ \mathbf{1}$
$15!$	$\Rightarrow$	$1\cdot 2^3+1\cdot 2^2+1\cdot 2^1+2^0$	$\Rightarrow$	$(15-\mathbf{4})$	$=$	$\mathbf{11}$	$>\ \mathbf{0}$
$16!$	$\Rightarrow$	2^4	$\Rightarrow$	$(16-\mathbf{1})$	$=$	$\mathbf{15}$	$>\ \mathbf{4}$
$17!$	$\Rightarrow$	2^4+2^0	$\Rightarrow$	$(17-\mathbf{2})$	$=$	$\mathbf{15}$	$>\ \mathbf{0}$
$18!$	$\Rightarrow$	2^4+2^1	$\Rightarrow$	$(18-\mathbf{2})$	$=$	$\mathbf{16}$	$>\ \mathbf{1}$
$19!$	$\Rightarrow$	$2^4+2^1+2^0$	$\Rightarrow$	$(19-\mathbf{3})$	$=$	$\mathbf{16}$	$>\ \mathbf{0}$
$20!$	$\Rightarrow$	2^4+2^2	$\Rightarrow$	$(20-\mathbf{2})$	$=$	$\mathbf{18}$	$>\ \mathbf{2}$
$21!$	$\Rightarrow$	$2^4+2^2+2^0$	$\Rightarrow$	$(21-\mathbf{3})$	$=$	$\mathbf{18}$	$>\ \mathbf{0}$
$22!$	$\Rightarrow$	$2^4+2^2+2^1$	$\Rightarrow$	$(22-\mathbf{3})$	$=$	$\mathbf{19}$	$>\ \mathbf{1}$
$23!$	$\Rightarrow$	$2^4+2^2+2^1+2^0$	$\Rightarrow$	$(23-\mathbf{4})$	$=$	$\mathbf{19}$	$>\ \mathbf{0}$
$24!$	$\Rightarrow$	2^4+2^3	$\Rightarrow$	$(24-\mathbf{2})$	$=$	$\mathbf{22}$	$>\ \mathbf{3}$
$25!$	$\Rightarrow$	$2^4+2^3+2^0$	$\Rightarrow$	$(25-\mathbf{3})$	$=$	$\mathbf{22}$	$>\ \mathbf{0}$
$26!$	$\Rightarrow$	$2^4+2^3+2^1$	$\Rightarrow$	$(26-\mathbf{3})$	$=$	$\mathbf{23}$	$>\ \mathbf{1}$
$27!$	$\Rightarrow$	$2^4+2^3+2^1+2^0$	$\Rightarrow$	$(27-\mathbf{4})$	$=$	$\mathbf{23}$	$>\ \mathbf{0}$
$28!$	$\Rightarrow$	$2^4+2^3+2^2$	$\Rightarrow$	$(28-\mathbf{3})$	$=$	$\mathbf{25}$	$>\ \mathbf{2}$
$29!$	$\Rightarrow$	$2^4+2^3+2^2+2^0$	$\Rightarrow$	$(29-\mathbf{4})$	$=$	$\mathbf{25}$	$>\ \mathbf{0}$
$30!$	$\Rightarrow$	$2^4+2^3+2^2+2^1$	$\Rightarrow$	$(30-\mathbf{4})$	$=$	$\mathbf{26}$	$>\ \mathbf{1}$
$31!$	$\Rightarrow$	$2^4+2^3+2^2+2^1+2^0$	$\Rightarrow$	$(31-\mathbf{5})$	$=$	$\mathbf{26}$	$>\ \mathbf{0}$
$32!$	$\Rightarrow$	2^5	$\Rightarrow$	$(32-\mathbf{1})$	$=$	$\mathbf{31}$	$>\ \mathbf{5}$
$33!$	$\Rightarrow$	2^5+2^0	$\Rightarrow$	$(33-\mathbf{2})$	$=$	$\mathbf{31}$	$>\ \mathbf{0}$

$\boldsymbol{P=3}$

				$A_3\downarrow$		$\mathbf{ord_3}\boldsymbol{n!}\downarrow$	$B_3\downarrow$
$0!$	$\Rightarrow$	$0\cdot 3^0$	$\Rightarrow$	$\frac{1}{2}\cdot(0-\mathbf{0})$	$=$	$\mathbf{0}$	$>\ \mathbf{0}$
$1!$	$\Rightarrow$	$1\cdot 3^0$	$\Rightarrow$	$\frac{1}{2}\cdot(1-\mathbf{1})$	$=$	$\mathbf{0}$	$>\ \mathbf{0}$
$2!$	$\Rightarrow$	$2\cdot 3^0$	$\Rightarrow$	$\frac{1}{2}\cdot(2-\mathbf{2})$	$=$	$\mathbf{0}$	$>\ \mathbf{0}$
$3!$	$\Rightarrow$	$1\cdot 3^1$	$\Rightarrow$	$\frac{1}{2}\cdot(3-\mathbf{1})$	$=$	$\mathbf{1}$	$>\ \mathbf{1}$

				$\mathbf{A_3}$ ↓		**ord$_3 n!$** ↓	$\mathbf{B_3}$ ↓
							> **0**
4!	⇒	$1\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(4-\mathbf{2})$	=	**1**	
							> **0**
5!	⇒	$1\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(5-\mathbf{3})$	=	**1**	
							> **1**
6!	⇒	$2\cdot3^1$	⇒	$\frac{1}{2}\cdot(6-\mathbf{2})$	=	**2**	
							> **0**
7!	⇒	$2\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(7-\mathbf{3})$	=	**2**	
							> **0**
8!	⇒	$2\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(8-\mathbf{4})$	=	**2**	
							> **2**
9!	⇒	$1\cdot3^2$	⇒	$\frac{1}{2}\cdot(9-\mathbf{1})$	=	**4**	
							> **0**
10!	⇒	$1\cdot3^2+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(10-\mathbf{2})$	=	**4**	
							> **0**
11!	⇒	$1\cdot3^2+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(11-\mathbf{3})$	=	**4**	
							> **1**
12!	⇒	$1\cdot3^2+1\cdot3^1$	⇒	$\frac{1}{2}\cdot(12-\mathbf{2})$	=	**5**	
							> **0**
13!	⇒	$1\cdot3^2+1\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(13-\mathbf{3})$	=	**5**	
							> **0**
14!	⇒	$1\cdot3^2+1\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(14-\mathbf{4})$	=	**5**	
							> **1**
15!	⇒	$1\cdot3^2+2\cdot3^1$	⇒	$\frac{1}{2}\cdot(15-\mathbf{3})$	=	**6**	
							> **0**
16!	⇒	$1\cdot3^2+2\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(16-\mathbf{4})$	=	**6**	
							> **0**
17!	⇒	$1\cdot3^2+2\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(17-\mathbf{5})$	=	**6**	
							> **2**
18!	⇒	$2\cdot3^2$	⇒	$\frac{1}{2}\cdot(18-\mathbf{2})$	=	**8**	
							> **0**
19!	⇒	$2\cdot3^2+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(19-\mathbf{3})$	=	**8**	
							> **0**
20!	⇒	$2\cdot3^2+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(20-\mathbf{4})$	=	**8**	
							> **1**
21!	⇒	$2\cdot3^2+1\cdot3^1$	⇒	$\frac{1}{2}\cdot(21-\mathbf{3})$	=	**9**	
							> **0**
22!	⇒	$2\cdot3^2+1\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(22-\mathbf{4})$	=	**9**	
							> **0**
23!	⇒	$2\cdot3^2+1\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(23-\mathbf{5})$	=	**9**	
							> **1**
24!	⇒	$2\cdot3^2+2\cdot3^1$	⇒	$\frac{1}{2}\cdot(24-\mathbf{4})$	=	**10**	
							> **0**
25!	⇒	$2\cdot3^2+2\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(25-\mathbf{5})$	=	**10**	
							> **0**
26!	⇒	$2\cdot3^2+2\cdot3^1+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(26-\mathbf{6})$	=	**10**	
							> **3**
27!	⇒	$1\cdot3^3$	⇒	$\frac{1}{2}\cdot(27-\mathbf{1})$	=	**13**	
							> **0**
28!	⇒	$1\cdot3^3+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(28-\mathbf{2})$	=	**13**	
							> **0**
29!	⇒	$1\cdot3^3+2\cdot3^0$	⇒	$\frac{1}{2}\cdot(29-\mathbf{3})$	=	**13**	
							> **1**
30!	⇒	$1\cdot3^3+1\cdot3^1$	⇒	$\frac{1}{2}\cdot(30-\mathbf{2})$	=	**14**	
							> **0**
31!	⇒	$1\cdot3^3+1\cdot3^1+1\cdot3^0$	⇒	$\frac{1}{2}\cdot(31-\mathbf{3})$	=	**14**	
							> **0**

				$A_3 \downarrow$		$\mathrm{ord}_3 n! \downarrow$	$B_3 \downarrow$
32!	⇒	$1 \cdot 3^3 + 1 \cdot 3^1 + 2 \cdot 3^0$	⇒	$\frac{1}{2} \cdot (32 - \mathbf{4})$	=	**14**	
							$>$ **1**
33!	⇒	$1 \cdot 3^3 + 2 \cdot 3^1$	⇒	$\frac{1}{2} \cdot (33 - \mathbf{3})$	=	**15**	
							$>$ **0**
34!	⇒	$1 \cdot 3^3 + 2 \cdot 3^1 + 1 \cdot 3^0$	⇒	$\frac{1}{2} \cdot (34 - \mathbf{4})$	=	**15**	
							$>$ **0**
35!	⇒	$1 \cdot 3^3 + 2 \cdot 3^1 + 2 \cdot 3^0$	⇒	$\frac{1}{2} \cdot (35 - \mathbf{5})$	=	**15**	

$P = 5$

				$A_5 \downarrow$		$\mathrm{ord}_5 n! \downarrow$	$B_5 \downarrow$
							$>$ **0**
0!	⇒	$0 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (0 - \mathbf{0})$	=	**0**	
							$>$ **0**
1!	⇒	$1 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (1 - \mathbf{1})$	=	**0**	
							$>$ **0**
2!	⇒	$2 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (2 - \mathbf{2})$	=	**0**	
							$>$ **0**
3!	⇒	$3 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (3 - \mathbf{3})$	=	**0**	
							$>$ **0**
4!	⇒	$4 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (4 - \mathbf{4})$	=	**0**	
							$>$ **1**
5!	⇒	$1 \cdot 5^1 + 0 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (5 - \mathbf{1})$	=	**1**	
							$>$ **0**
6!	⇒	$1 \cdot 5^1 + 1 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (6 - \mathbf{2})$	=	**1**	
							$>$ **0**
7!	⇒	$1 \cdot 5^1 + 2 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (7 - \mathbf{3})$	=	**1**	
							$>$ **0**
8!	⇒	$1 \cdot 5^1 + 3 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (8 - \mathbf{4})$	=	**1**	
							$>$ **0**
9!	⇒	$1 \cdot 5^1 + 4 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (9 - \mathbf{5})$	=	**1**	
							$>$ **1**
10!	⇒	$2 \cdot 5^1 + 0 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (10 - \mathbf{2})$	=	**2**	
							$>$ **0**
11!	⇒	$2 \cdot 5^1 + 1 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (11 - \mathbf{3})$	=	**2**	
							$>$ **0**
12!	⇒	$2 \cdot 5^1 + 2 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (12 - \mathbf{4})$	=	**2**	
							$>$ **0**
13!	⇒	$2 \cdot 5^1 + 3 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (13 - \mathbf{5})$	=	**2**	
							$>$ **0**
14!	⇒	$2 \cdot 5^1 + 4 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (14 - \mathbf{6})$	=	**2**	
							$>$ **1**
15!	⇒	$3 \cdot 5^1$	⇒	$\frac{1}{4} \cdot (15 - \mathbf{3})$	=	**3**	
							$>$ **0**
16!	⇒	$3 \cdot 5^1 + 1 \cdot 5^0$	⇒	$\frac{1}{4} \cdot (16 - \mathbf{4})$	=	**3**	

				$\boldsymbol{A_5}\downarrow$		$\mathbf{ord}_5 \boldsymbol{n!}\downarrow$	$\boldsymbol{B_5}\downarrow$
17!	⇒	$3\cdot 5^1 + 2\cdot 5^0$	⇒	$\frac{1}{4}\cdot(17-\mathbf{5})$	=	$\mathbf{3}$	$> \mathbf{0}$
18!	⇒	$3\cdot 5^1 + 3\cdot 5^0$	⇒	$\frac{1}{4}\cdot(18-\mathbf{6})$	=	$\mathbf{3}$	$> \mathbf{0}$
19!	⇒	$3\cdot 5^1 + 4\cdot 5^0$	⇒	$\frac{1}{4}\cdot(19-\mathbf{7})$	=	$\mathbf{3}$	$> \mathbf{0}$
20!	⇒	$4\cdot 5^1$	⇒	$\frac{1}{4}\cdot(20-\mathbf{4})$	=	$\mathbf{4}$	$> \mathbf{1}$
21!	⇒	$4\cdot 5^1 + 1\cdot 5^0$	⇒	$\frac{1}{4}\cdot(21-\mathbf{5})$	=	$\mathbf{4}$	$> \mathbf{0}$
22!	⇒	$4\cdot 5^1 + 2\cdot 5^0$	⇒	$\frac{1}{4}\cdot(22-\mathbf{6})$	=	$\mathbf{4}$	$> \mathbf{0}$
23!	⇒	$4\cdot 5^1 + 3\cdot 5^0$	⇒	$\frac{1}{4}\cdot(23-\mathbf{7})$	=	$\mathbf{4}$	$> \mathbf{0}$
24!	⇒	$4\cdot 5^1 + 4\cdot 5^0$	⇒	$\frac{1}{4}\cdot(24-\mathbf{8})$	=	$\mathbf{4}$	$> \mathbf{0}$
25!	⇒	$1\cdot 5^2$	⇒	$\frac{1}{4}\cdot(25-\mathbf{1})$	=	$\mathbf{6}$	$> \mathbf{2}$
26!	⇒	$1\cdot 5^2 + 1\cdot 5^0$	⇒	$\frac{1}{4}\cdot(26-\mathbf{2})$	=	$\mathbf{6}$	$> \mathbf{0}$
27!	⇒	$1\cdot 5^2 + 2\cdot 5^0$	⇒	$\frac{1}{4}\cdot(27-\mathbf{3})$	=	$\mathbf{6}$	$> \mathbf{0}$
28!	⇒	$1\cdot 5^2 + 3\cdot 5^0$	⇒	$\frac{1}{4}\cdot(28-\mathbf{4})$	=	$\mathbf{6}$	$> \mathbf{0}$
29!	⇒	$1\cdot 5^2 + 4\cdot 5^0$	⇒	$\frac{1}{4}\cdot(29-\mathbf{5})$	=	$\mathbf{6}$	$> \mathbf{0}$
30!	⇒	$1\cdot 5^2 + 1\cdot 5^1$	⇒	$\frac{1}{4}\cdot(30-\mathbf{2})$	=	$\mathbf{7}$	$> \mathbf{1}$
31!	⇒	$1\cdot 5^2 + 1\cdot 5^1 + 1\cdot 5^0$	⇒	$\frac{1}{4}\cdot(31-\mathbf{3})$	=	$\mathbf{7}$	$> \mathbf{0}$
32!	⇒	$1\cdot 5^2 + 1\cdot 5^1 + 2\cdot 5^0$	⇒	$\frac{1}{4}\cdot(32-\mathbf{4})$	=	$\mathbf{7}$	$> \mathbf{0}$
33!	⇒	$1\cdot 5^2 + 1\cdot 5^1 + 3\cdot 5^0$	⇒	$\frac{1}{4}\cdot(33-\mathbf{5})$	=	$\mathbf{7}$	$> \mathbf{0}$
34!	⇒	$1\cdot 5^2 + 1\cdot 5^1 + 4\cdot 5^0$	⇒	$\frac{1}{4}\cdot(34-\mathbf{6})$	=	$\mathbf{7}$	$> \mathbf{0}$
35!	⇒	$1\cdot 5^2 + 2\cdot 5^1$	⇒	$\frac{1}{4}\cdot(35-\mathbf{3})$	=	$\mathbf{8}$	$> \mathbf{1}$
36!	⇒	$1\cdot 5^2 + 2\cdot 5^1 + 1\cdot 5^0$	⇒	$\frac{1}{4}\cdot(36-\mathbf{4})$	=	$\mathbf{8}$	$> \mathbf{0}$
37!	⇒	$1\cdot 5^2 + 2\cdot 5^1 + 2\cdot 5^0$	⇒	$\frac{1}{4}\cdot(37-\mathbf{5})$	=	$\mathbf{8}$	$> \mathbf{0}$
38!	⇒	$1\cdot 5^2 + 2\cdot 5^1 + 3\cdot 5^0$	⇒	$\frac{1}{4}\cdot(38-\mathbf{6})$	=	$\mathbf{8}$	$> \mathbf{0}$
39!	⇒	$1\cdot 5^2 + 2\cdot 5^1 + 4\cdot 5^0$	⇒	$\frac{1}{4}\cdot(39-\mathbf{7})$	=	$\mathbf{8}$	$> \mathbf{0}$
40!	⇒	$1\cdot 5^2 + 3\cdot 5^1$	⇒	$\frac{1}{4}\cdot(40-\mathbf{4})$	=	$\mathbf{9}$	$> \mathbf{1}$

etc.

The numerical sequence with the members $\big(\alpha_{p,(r_n)}\big)$ with the members $\alpha_{p,n}$, consisting of sums of coefficients $\sum a_x$, which correspond to $n!$ for the given prime number p of the single given positive integer $0 \le n \le \infty$, is the numerical sequence $\big(A_{p,(r_n)}\big)$ for the given prime number p from a certain numerical sequence $(r_n) =$

$(0,1,2,...,n)$ of the members $0 \leq n \leq \infty$.[1] That is:

$$\left(A_{p,(r_n)}\right) = \left(\alpha_{p,(r_n)}\right) = \left(\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3}, ..., \alpha_{p,n}\right). \tag{2.1}$$

One writes the derivation of Chebyshev's formula to show how to find the sum of coefficients $\sum a_x$ for the certain positive integer $n = 0,1,2,...,\infty$ for the given prime number p:

$$\begin{aligned}
&n = a_0 + a_1 \cdot p + a_2 \cdot p^2 + ... + a_x \cdot p^x,\\
&\alpha_{p,n} = S_n = \sum_{t=0}^{p^x \leq n} a_t = a_0 + a_1 + a_2 + ... + a_x,\\
&0 \leq a_t \leq p-1,\\
&ord_p n! = \frac{n - S_n}{p-1}, \qquad (2.2)
\end{aligned}$$

or

$$\begin{aligned}
&ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right) =\\
&= \left(\frac{1}{p-1} \cdot \left(0 - \alpha_{p,0}\right), \frac{1}{p-1} \cdot \left(1 - \alpha_{p,1}\right), \frac{1}{p-1} \cdot \left(2 - \alpha_{p,2}\right), ..., \frac{1}{p-1} \cdot \left(n - \alpha_{p,n}\right)\right) =\\
&= \frac{1}{p-1} \cdot \left((0,1,2,...,n) - \left(\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}, ..., \alpha_{p,n}\right)\right) = \frac{1}{p-1} \cdot \left(\left(r_n\right) - \left(\alpha_{p,(r_n)}\right)\right) =\\
&= \frac{1}{p-1} \cdot \left(\left(r_n\right) - \left(A_{p,(r_n)}\right)\right),
\end{aligned}$$

(2.3)

or

$$\left(r_{n!}\right) = \prod_{p=2}^{p \leq n} p^{\frac{1}{p-1}} \cdot \left(\left(r_n\right) - \left(A_{p,(r_n)}\right)\right), \tag{2.4}$$

at

$$\begin{aligned}
&p \geq 2,\\
&0 \leq n \leq \infty,\\
&\left(r_n\right) = \left(0, 1, 2, \ldots, n\right),\\
&\left(r_{n!}\right) = \left(0!, 1!, 2!, \ldots, n!\right),\\
&\left(A_{p,(r_n)}\right) = \left(\alpha_{p,(r_n)}\right) = \left(\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}, \ldots, \alpha_{p,n}\right).
\end{aligned}$$

Below is shown how the numerical sequence $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n) looks in its development. Then the law of formation of the numerical sequence $\left(A_{p,(r_n)}\right)$ for the given prime number p will be written, with which any member of the numerical sequence can be determined.

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[1] cf. Maraev, Said - Development of Chebyshev's formula for prime factorization of $n!$ as a product of the prime numbers p with the corresponding exponents, and the derivation of Chebyshevs formula; in a forthcoming paper.

The numerical sequences $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence $(r_n) = (0, 1, 2, ..., n)$ and the number 1 are:

$$\begin{aligned}
\left(A_{1,(r_n)}\right) &= (0,1,2,3,4,5,...)\,, \\
\left(A_{2,(r_n)}\right) &= (0,1,1,2,\ 1,2,2,3,\ 1,2,2,3,\ 2,3,3,4,...)\,, \\
\left(A_{3,(r_n)}\right) &= (0,1,2,\ 1,2,3,\ 2,3,4,\ 1,2,3,\ 2,3,4,\ 3,4,5,...)\,, \\
\left(A_{5,(r_n)}\right) &= (0,1,2,3,4,\ 1,2,3,4,5,\ 2,3,4,5,6,\ 3,4,5,6,7,\ 4,5,6,7,8,...)\,, \\
&\textit{etc.}
\end{aligned}$$

2.1. **The numerical sequences $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence $(r_n) = (0, 1, 2, ..., n)$ in the development.**

$\underline{p = 2}$,

$$\begin{aligned}
&(0) \quad 2^0 \triangleq \overset{(2^0-1).}{(0)}, \\
&(1) \quad 2^1 \triangleq (\overset{0,}{0}, \overset{(2^1-1).}{1}), \\
&(2) \quad 2^2 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{1}, \overset{(2^2-1).}{2}), \\
&(3) \quad 2^3 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{1}, \overset{3,}{2}, \overset{...,}{1}, 2, 2, \overset{(2^3-1).}{3,}) \\
&(4) \quad 2^4 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{1}, \overset{3,}{2}, \overset{...,}{1}, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, \overset{(2^4-1).}{4}), \\
&\quad \text{etc.}
\end{aligned}$$

$\underline{p = 3}$,

$$\begin{aligned}
&(0) \quad 3^0 \triangleq \overset{(3^0-1).}{(0)}, \\
&(1) \quad 3^1 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{(3^1-1).}{2}), \\
&(2) \quad 3^2 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{...,}{1}, 2, 3, 2, 3, \overset{(3^2-1).}{4}), \\
&(3) \quad 3^3 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{...,}{1}, 2, 3, 2, 3, 4, 1, 2, 3, 2, 3, 4, 3, 4, 5, 2, 3, 4, 3, 4, 5, 4, \overset{...,}{5}, \overset{(3^3-1).}{6}), \\
&\quad \text{etc.}
\end{aligned}$$

$\underline{p = 5}$,

$$\begin{aligned}
&(0) \quad 5^0 \triangleq \overset{(5^0-1).}{(0)}, \\
&(1) \quad 5^1 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{...,}{3}, \overset{(5^1-1).}{4}), \\
&(2) \quad 5^2 \triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{3,}{3}, \overset{...}{4},\ 1, 2, 3, 4, 5,\ 2, 3, 4, 5, 6,\ 3, 4, 5, 6, 7,\ 4, 5, 6, \overset{...,}{7}, \overset{(5^2-1).}{8}),
\end{aligned}$$

$$(3)\qquad 5^3 \triangleq \Big(\begin{array}{l} \overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{3,}{3}, \overset{\ldots}{4},\ \ 1,2,3,4,5,\ \ 2,3,4,5,6,\ \ 3,4,5,6,7,\ \ 4,5,6,7,8, \\ 1,2,3,4,5,\ \ 2,3,4,5,6,\ \ 3,4,5,6,7,\ \ 4,5,6,7,8,\ \ 5,6,7,8,9, \\ 2,3,4,5,6,\ \ 3,4,5,6,7,\ \ 4,5,6,7,8,\ \ 5,6,7,8,9,\ \ 6,7,8,9,10, \\ 3,4,5,6,7,\ \ 4,5,6,7,8,\ \ 5,6,7,8,9,\ \ 6,7,8,9,10,\ \ 7,8,9,10,11, \\ 4,5,6,7,8,\ \ 5,6,7,8,9,\ \ 6,7,8,9,10,\ \ 7,8,9,10,11,\ \ 8,9,10,\overset{\ldots,}{11},\ \ \overset{(5^3-1).}{12} \end{array} \Big),$$

etc.

Every prime number with the certain exponent $\{k = p^x\}$ left of the numerical sequence $\left(A_{p,(r_n)}\right)$ is the monitoring number k. With the help of the monitoring number one can find out the size of the numerical sequence $\left(A_{p,(r_n)}\right)$, i.e. of how many members of the numerical sequence $\alpha_{p,n}$ the numerical sequence $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n) consists. The number of members $\alpha_{p,n}$ of the numerical sequence $\left(A_{p,(r_n)}\right)$ equals the monitoring number k.

2.2. Law of formation for the numerical sequence $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n).

(1) $\left(A_{p,(r_n)}\right)$ is a numerical sequence in which every member $\alpha_{p,n}$ is a positive integer from 0 up to $(p-1)\cdot\eta$, where $0 \leq \eta \leq \infty$:

$$0 \leq \alpha_{p,n} \leq (p-1)\cdot\eta. \tag{2.5}$$

(2) (a) Every single member $\alpha_{p,n}$ of the numerical sequence $\left(A_{p,(r_n)}\right)$ corresponds to a certain number, i.e.

$$\left(A_{p,(r_n)}\right) = \left(\alpha_{p,(r_n)}\right) = \left(\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}, \alpha_{p,3}, ..., \alpha_{p,(p^\eta-1)}\right), \tag{2.6}$$

$$Nr.\left(\alpha_{p,n}\right) = n = 0,1,2,...,\left(p^\eta - 1\right),$$

where

$$0 \leq \eta \leq \infty.$$

(b) The numerical sequence $\left(A_{p,(r_n)}\right)$ is the same numerical sequence consisting of the members $\left(\alpha_{p,(r_n)}\right)$, which consists of the single members $\alpha_{p,n}$, i.e.

$$\left(A_{p,(r_n)}\right) = \left(\alpha_{p,(r_n)}\right) = \left(\alpha_{p,0}, \alpha_{p,1}, \alpha_{p,2}, ..., \alpha_{p,(p^\eta-1)}\right), \tag{2.7}$$

$$\left(Nr.Nr.\left(\alpha_{p,(r_n)}\right)\right) = (r_n) = \left(0,1,2,...,\left(p^\eta-1\right)\right),$$

where

$$\begin{aligned} &0 \leq \eta \leq \infty, \\ &2 \leq p \leq \infty, \\ &0 \leq n \leq \left(p^\eta - 1\right), \\ &(r_n) = \left(0,1,2,...,\left(p^\eta-1\right)\right). \end{aligned}$$

Here (r_n) is the numerical sequence of the members n.

(3) The numerical sequence $\left(A_{p,(r_n)}\right)$ corresponds to the monitoring number $(k = p^\eta)$, i.e.

$$\left(A_{p,(r_n)}\right) \triangleq \left(k = p^\eta\right). \tag{2.8}$$

(4) The sum of all members $\alpha_{p,n}$ of the numerical sequence $\left(A_{p,(r_n)}\right)$ equals

$$\sum_{n=0}^{p^\eta-1} \alpha_{p,n} = \frac{1}{2} \cdot (p-1) \cdot \eta \cdot p^\eta. \tag{2.9}$$

(5) The last member $\alpha_{p,(p^\eta-1)}$ of the numerical sequence $\left(A_{p,(r_n)}\right)$ equals

$$\alpha_{p,(p^\eta-1)} = (p-1) \cdot \eta. \tag{2.10}$$

(6) The quantity $Q_{\left(A_{p,(r_n)}\right)}$ of all members $\alpha_{p,n}$ of the numerical sequence $\left(A_{p,(r_n)}\right)$ equals

$$Q_{\left(A_{p,(r_n)}\right)} = p^\eta, \tag{2.11}$$

where

$$2 \le p \le \infty,$$
$$0 \le \eta \le \infty.$$

(7) Variation I. The numerical sequence $\left(A_{p,(r_n)}\right)$ has to be considered a sum of numerical sequences $(a) = (0, 1, 2, ..., (p-1))$:

$$\begin{aligned} (\overset{\triangleq p^\eta}{A}{}_{p,(r_n)}) &= ((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + ... + (\overset{\eta}{a})) = \\ &= ((0, 1, 2, \overset{1,}{...}, (p-1)) + (0, 1, 2, \overset{2,}{...}, (p-1)) + \overset{...,}{...} + (0, 1, 2, \overset{\eta}{...}, (p-1))), \end{aligned} \tag{2.12}$$

$$ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., (p^\eta - 1)!\right) = \frac{1}{p-1} \cdot \left((r_n) - \left(A_{p,(r_n)}\right)\right) \tag{2.13}$$

and

$$1 \le \eta \le \infty,$$
$$2 \le p \le \infty,$$
$$0 \le n \le (p^\eta - 1),$$
$$(a) = (0, 1, 2, ..., (p-1)),$$
$$(r_n) = (0, 1, 2, ..., (p^\eta - 1)).$$

Example 1. Variation I.

$$p = 2,$$
$$\eta = 3,$$
$$0 \le n \le \left(2^3 - 1\right),$$
$$(r_n) = \left(0, 1, 2, ..., \left(2^3 - 1\right)\right),$$
$$(r_{n!}) = (0!, 1!, 2!, ..., n!).$$

Then

$$\begin{aligned}
(\overset{\triangleq 2^3}{A}_{2,(0,1,2,...,(2^3-1))}) &= ((\overset{1,}{(a)}+\overset{2,}{(a)}+\overset{3,}{(a)}) = \\
&= ((\overset{1,}{(0,1)}+\overset{2,}{(0,1)}+\overset{3,}{(0,1)}) = \\
&= (\ \overset{0,}{0},\overset{1,}{1},\overset{2,}{1},\overset{...,}{2},1,2,2,\overset{(2^3-1).}{3}\ ).
\end{aligned}$$

And

$$\begin{aligned}
&ord_2\left(0!,1!,2!,...,\left(2^3-1\right)!\right) = \\
= \ &\frac{1}{2-1}\cdot((0,1,2,...,(2^3-1)) - (\overset{0,}{0},\overset{1,}{1},\overset{2,}{1},\overset{...,}{2},1,2,2,\overset{(2^3-1).}{3}\ )) = \\
= \ &(\overset{0,}{0}\ ,\overset{1,}{0},\overset{2,}{1},\overset{...,}{1},3,3,4,\overset{(2^3-1).}{4}\ ).
\end{aligned}$$

(8) Variation II.a. The sum of the numerical sequences $\left(A_{p,\left(r_{n_i}\right)}\right)$, where $1 \leq i \leq \infty$, adds up to the complete numerical sequence $\left(A_{p,(r_n)}\right)$. While every numerical sequence $\left(A_{p,\left(r_{n_i}\right)}\right)$ possesses a corresponding monitoring number $(k = p^{x_i})$, the complete numerical sequence $\left(A_{p,(r_n)}\right)$ itself possesses a corresponding monitoring number $(k = p^{x_1+x_2+x_3+...+x_i})$, which is the given prime number p, consisting of the sum of all exponents corresponding to every monitoring number of every numerical sequence $\left(A_{p,\left(r_{n_i}\right)}\right)$:

$$((\overset{\triangleq p^{x_1}}{A}_{p,\left(r_{n_1}\right)}) + (\overset{\triangleq p^{x_2}}{A}_{p,\left(r_{n_2}\right)}) + (\overset{\triangleq p^{x_3}\ ...}{A}_{p,\left(r_{n_3}\right)}) +... + (\overset{\triangleq p^{x_i}}{A}_{p,\left(r_{n_i}\right)})) = \overset{\triangleq p^{x_1+x_2+x_3+...+x_i}}{(\quad A_{p,(r_n)}\quad)}. \tag{2.14}$$

With

$$\begin{aligned}
&(\overset{\triangleq p^{x_1}}{A}_{p,\left(r_{n_1}\right)}) = ((\overset{1,}{(a)}+\overset{2,}{(a)}+\overset{3,...,}{(a)}+...+\overset{x_1}{(a)}) = \\
&= ((0,1,2,\overset{1,}{...},(p-1)) + (0,1,2,\overset{2,...,}{...},(p-1)) +... + (0,1,2,\overset{x_1}{...},(p-1))), \\
&(\overset{\triangleq p^{x_2}}{A}_{p,\left(r_{n_2}\right)}) = ((\overset{1,}{(a)}+\overset{2,}{(a)}+\overset{3,...,}{(a)}+...+\overset{x_2}{(a)}) = \\
&= ((0,1,2,\overset{1,}{...},(p-1)) + (0,1,2,\overset{2,...,}{...},(p-1)) +... + (0,1,2,\overset{x_2}{...},(p-1))), \\
&\qquad\qquad\qquad\qquad\vdots \\
&(\overset{\triangleq p^{x_i}}{A}_{p,\left(r_{n_i}\right)}) = ((\overset{1,}{(a)}+\overset{2,}{(a)}+\overset{3,...,}{(a)}+...+\overset{x_i}{(a)}) = \\
&= ((0,1,2,\overset{1,}{...},(p-1)) + (0,1,2,\overset{2,...,}{...},(p-1)) +... + (0,1,2,\overset{x_i}{...},(p-1))).
\end{aligned} \tag{2.15}$$

Then

$$\begin{aligned} ord_p\left(r_{n!}\right) &= \left(0!,1!,2!,...,\left(p^{x_1+x_2+x_3+...+x_i}-1\right)!\right) = \\ &= \frac{1}{p-1}\cdot\left(\left(r_n\right)-\left(A_{p,(r_n)}\right)\right) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &1 \le x_i \le \infty, \\ &2 \le p \le \infty, \\ &0 \le n \le \left(p^{x_i}-1\right), \\ &i = 1,2,3,... \\ &(a) = (0,1,2,...,(p-1)), \\ &\left(r_{n_i}\right) = \left(0,1,2,...,\left(p^{x_i}-1\right)\right), \\ &\left(r_n\right) = \left(0,1,2,...,\left(p^{x_1+x_2+x_3+...+x_i}-1\right)\right). \end{aligned}$$

Example 2. Variation II.a.

$$\begin{aligned} p &= 2, \\ x_1 &= 1, \\ x_2 &= 2, \\ x_3 &= 3. \end{aligned}$$

Then

$$\begin{aligned} &(\overset{\triangleq 2^1}{(0,1)} + \overset{\triangleq 2^2}{(0,1,1,2)} + \overset{\triangleq 2^3}{(0,1,1,2,\quad 1,2,2,3)}) = \\ &\qquad\qquad\qquad \triangleq 2^6 \\ = \ &(\overset{0,}{0},\overset{1,}{1},\overset{2,}{1},\overset{3,}{2},\quad \overset{\dots}{1},2,2,3,\quad 1,2,2,3,\quad 2,3,3,4, \\ &\ 1,2,2,3,\quad 2,3,3,4,\quad 2,3,3,4,\quad 3,4,4,5, \\ &\ 1,2,2,3,\quad 2,3,3,4,\quad 2,3,3,4,\quad 3,4,4,5, \\ &\ 2,3,3,4,\quad 3,4,4,5,\quad 3,4,4,5,\quad 4,5,\overset{\dots,}{5},\overset{(2^6-1).}{6}). \end{aligned}$$

Variation II.b. The sum of equal numerical sequences $\left(A_{p,\left(r_{n_1}\right)}\right)$ adds up to the complete numerical sequence $\left(A_{p,(r_n)}\right)$. The number of these equal numerical sequences $\left(A_{p,\left(r_{n_1}\right)}\right)$ is denoted as the quantity y. Each of these equal numerical sequences $\left(A_{p,\left(r_{n_1}\right)}\right)$ possesses a monitoring number $(k_1 = p^{x_1})$. Then the complete numerical sequence $\left(A_{p,(r_n)}\right)$ possesses a corresponding monitoring number $(k = p^{x_1\cdot y})$. All this is expressed as:

$$((\underset{1,}{\overset{\triangleq p^{x_1}}{A_{p,\left(r_{n_1}\right)}}}) + (\underset{2,\dots,}{\overset{\triangleq p^{x_1}\ \dots}{A_{p,\left(r_{n_1}\right)}}}) + ... + (\underset{y}{\overset{\triangleq p^{x_1}}{A_{p,\left(r_{n_1}\right)}}})) = (\overset{\triangleq p^{x_1\cdot y}}{A_{p,(r_n)}}). \tag{2.17}$$

Where

$$\Big(\overset{\triangleq p^{x_1}}{A}_{p,(r_{n_1})}\Big) = \Big(\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,\ldots,}{(a)} + \ldots + \overset{x_1}{(a)}\Big) =$$

$$= \Big((0,1,2,\overset{1,}{\ldots},(p-1)) + (0,1,2,\overset{2,\ldots,}{\ldots},(p-1)) + \ldots + (0,1,2,\overset{x_1}{\ldots},(p-1))\Big), \tag{2.18}$$

$$\begin{aligned} ord_p\,(r_{n!}) &= ord_p\,(0!,1!,2!,\ldots,(p^{x_l\cdot y}-1)!) = \\ &= \frac{1}{p-1}\cdot\big((r_n) - \big(A_{p,(r_n)}\big)\big) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &1 \le x_1 \le \infty, \\ &2 \le p \le \infty, \\ &0 \le n \le (p^{x_1}-1)\,, \\ &(a) = (0,1,2,\ldots,(p-1))\,, \\ &(r_{n_1}) = (0,1,2,\ldots,(p^{x_1}-1))\,, \\ &(r_n) = (0,1,2,\ldots,(p^{x_1\cdot y}-1))\,. \end{aligned}$$

Example 3. Variation II.b.

$$\begin{aligned} &p = 2, \\ &x_1 = 2, \\ &y = 3, \\ &(r_{n_1}) = \big(0,1,2,\ldots,\big(2^2-1\big)\big)\,, \\ &(r_n) = \big(0,1,2,\ldots,\big(2^{2\cdot 3}-1\big)\big)\,. \end{aligned}$$

Therefore

$$\Big((0,\underset{1,}{\overset{\triangleq 2^2}{1}},1,2) + (0,\underset{2,}{\overset{\triangleq 2^2}{1}},1,2) + (0,\underset{3.}{\overset{\triangleq 2^2}{1}},1,2)\Big) =$$

$$\begin{array}{llll} \Big(\overset{0,}{0},\overset{1,}{1},\overset{2,}{1},\overset{3,}{2}, & \overset{\overset{\triangleq 2^6}{\ldots}}{1},2,2,3 & 1,2,2,3, & 2,3,3,4, \\ 1,2,2,3, & 2,3,3,4, & 2,3,3,4, & 3,4,4,5, \\ 1,2,2,3, & 2,3,3,4, & 2,3,3,4, & 3,4,4,5, \\ 2,3,3,4, & 3,4,4,5, & 3,4,4,5, & 4,5,\overset{\ldots,}{5},\overset{(2^6-1).}{6}\Big). \end{array}$$

Variation II.c. It is more convenient to use step by step the following development of the numerical sequence $\big(A_{p,(r_n)}\big)$ for the given prime number p from a certain numerical sequence (r_n):

$$\Big(\big(\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{n_1})}\big) + \big(\underset{2,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{n_1})}\big) + \big(\underset{3,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{n_1})}\big) + \underset{\ldots,}{\overset{\ldots}{\ldots}} + \big(\underset{p}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,(r_{n_1})}\big)\Big) = \big(\overset{\triangleq p^{p^{\eta+1}}}{A}_{p,(r_{n_2})}\big), \tag{2.20}$$

$$\begin{aligned}
&(\overset{\triangleq p^{p^{\eta}}}{A}{}_{p,\left(r_{n_1}\right)}) = ((\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,\dots,}{(a)} + \dots + \overset{p^{\eta}}{(a)}) = \\
&= ((0,1,2,\overset{1,}{\dots},(p-1)) + (0,1,2,\overset{2,\dots,}{\dots},(p-1)) + \dots + (0,1,2,\overset{p^{\eta}}{\dots},(p-1))) = \\
&= (\overset{0,}{0}, \overset{1,}{1}, \overset{2,\dots,}{2}, \dots, (p-1), \dots, \overset{\left(p^{p^{\eta}}-1\right)}{p^{\eta}\cdot(p-1)}),
\end{aligned}\tag{2.21}$$

then

$$\begin{aligned}
ord_p\left(r_{(n_2)!}\right) &= ord_p\left(0!,1!,2!,\dots,\left(p^{p^{\eta+1}}-1\right)!\right) = \\
&= \frac{1}{p-1}\cdot\left(\left(r_{n_2}\right)-\left(A_{p,\left(r_{n_2}\right)}\right)\right) = \\
&= \frac{1}{p-1}\cdot\left(\left(0,1,2,\dots,\left(p^{p^{\eta+1}}-1\right)\right)-\right. \\
&\quad \left.-\left(0,1,2,\dots,\left(p^{\eta+1}\cdot(p-1)\right)\right)\right),
\end{aligned}\tag{2.22}$$

if $0\le n_1\le\left(p^{p^{\eta}}-1\right)$, $0\le n_2\le\left(p^{p^{\eta+1}}-1\right)$, $(a)=(0,1,2,\dots,(p-1))$,
$\left(r_{n_1}\right)=\left(0,1,2,\dots,\left(p^{p^{\eta}}-1\right)\right)$, $\alpha_{p,\left(p^{p^{\eta}}-1\right)}=p^{\eta}\cdot(p-1)$,
$Nr.\left(\alpha_{p,\left(p^{p^{\eta}}-1\right)}\right)=p^{p^{\eta}}-1$, $\left(r_{n_2}\right)=\left(0,1,2,\dots,\left(p^{p^{\eta+1}}-1\right)\right)$,
$\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}=p^{\eta+1}\cdot(p-1)$, $Nr.\left(\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}\right)=p^{p^{\eta+1}}-1$,
$0\le\eta\le\infty$, $p\ge 2$, $\alpha_{p,p^{\eta}}=1$, $\mathrm{Nr}\left(\alpha_{p,0}\right)=0$, $\alpha_{p,0}=0$.

Example 4. Variation II.c
$p=3$, $\eta=1$, $\left(r_{n_1}\right)=\left(0,1,2,\dots,\left(3^{3^1}-1\right)\right)$,
$\left(r_{n_2}\right)=\left(0,1,2,\dots,\left(3^{3^{1+1}}-1\right)\right)$, $(a)=(0,1,2)$,

$$(\overset{\triangleq 3^{3^1}}{A_{3,\left(r_{n_1}\right)}}) = \underset{1,}{\overset{\triangleq 3^{3^0}}{(0,1,2)}} + \underset{2,}{\overset{\triangleq 3^{3^0}}{(0,1,2)}} + \underset{3.}{\overset{\triangleq 3^{3^0}}{(0,1,2)}} =$$

$$= (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\dots}{1}, 2,3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4, \overset{\dots,}{5}, \overset{\left(3^{3^1}-1\right).}{6}).$$

Then

$$((\underset{1,}{\overset{\triangleq 3^{3^1}}{A_{3,\left(r_{n_1}\right)}}}) + (\underset{2,}{\overset{\triangleq 3^{3^1}}{A_{3,\left(r_{n_1}\right)}}}) + (\underset{3.}{\overset{\triangleq 3^{3^1}}{A_{3,\left(r_{n_1}\right)}}})) = (\overset{\triangleq 3^{3^{1+1}}}{A_{3,\left(r_{n_2}\right)}}) =$$

$$\begin{aligned}
= \quad & (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\dots}{1}, 2,3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4, \overset{\dots,}{5}, \overset{\left(3^{3^1}-1\right).}{6}) + \\
& +(\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\dots}{1}, 2,3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4, \overset{\dots,}{5}, \overset{\left(3^{3^1}-1\right).}{6}) + \\
& +(\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\dots}{1}, 2,3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4, \overset{\dots,}{5}, \overset{\left(3^{3^1}-1\right).}{6}) =
\end{aligned}$$

$$= \left(\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\dots}{1}, 2, 3, 2, 3, 4, \dots\dots\dots\dots\dots\dots\dots\dots\dots, 16, \overset{\dots,}{17}, \overset{\left(3^{3^{2}}-1\right).}{18}\right).$$

Then

$$ord_3\left(r_{(n_2)!}\right) =$$

$$= ord_3\left(0!, 1!, 2!, ..., \left(3^{3^{1+1}}-1\right)!\right) = \frac{1}{p-1}\cdot\left(\left(r_{n_2}\right) - \left(A_{3,\left(r_{n_2}\right)}\right)\right) =$$

$$= \frac{1}{3-1}\cdot\left(\left(0, 1, 2, ..., \left(3^{3^{1+1}}-1\right)\right) - \left(0, 1, 2, 1, 2, 3, ..., \left(3^{1+1}\cdot(3-1)\right)\right)\right) =$$

$$= \left(\overset{0,}{0}, \overset{1,}{0}, \overset{2,}{0}, \overset{\dots}{1}, 1, 1, 2, 2, 2, 4, 4, 4, 5, 5, 5, 6, 6, 6, 8, 8, 8, 9, 9, 9, 10, \overset{\dots,}{10}, \overset{\left(3^{3^{1}}-1\right),}{10}\right.,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\left.\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots 9832, \overset{\dots,}{9832}, \overset{\left(3^{3^{1+1}}-1\right).}{9832}\right).$$

Remark 1. (1) The sum of several numerical sequences $\left(A_{p,(r_n)}\right)$. We will look at the case of a sum of two numerical sequences $\left(A_{p,(r_n)}\right)$ as an example because it is transferable to other cases:

$$\begin{aligned} &((a,b,c)+(a,b,c)) = \\ =&\left((a+a),(a+b),(a+c),\right. \\ &(b+a),(b+b),(b+c), \\ &\left.(c+a),(c+b),(c+c)\right). \end{aligned}$$

(2) The difference between the numerical sequence of numbers and the numerical sequence $\left(A_{p,(r_n)}\right)$ is the numerical sequence consisting of those numbers one has obtained as a result after having subtracted every number of the latter numerical sequence from every number of the first numerical sequence according to the order of every member in these numerical sequences.
E.g.:
$p = 3$,
$(0,1,2,3,4,5,\dots,n) - (0,1,2,1,2,3,\dots,(n-2\cdot t)) =$
$= (0,0,0,2,2,2,\dots,2\cdot t)$.
Where
$t = 0,1,2,\dots,\infty$,
$2\cdot t < n$.

(3) The division of one numerical sequence by a certain positive integer t means to divide every single member of this numerical sequence separately in the correct order by the positive integer t. Then one obtains a new numerical sequence as a result.
E.g.
$(0,0,0,2,2,2,\dots,2\cdot t) : 2 = (0,0,0,1,1,1,\dots,t)$.

(4) The multiplication of a numerical sequence with a certain positive integer t means to multiply every member in the correct order separately by t and thus obtaining a new numerical sequence.
$(0,0,0,1,1,1,\dots,t)\cdot 2 = (0,0,0,2,2,2,\dots,2\cdot t)$.

Remark 2. In the numerical sequence $(r_n) = (0, 1, 2, \ldots, n)$ the prime number $p \geq 2, p \in (r_n)$ is repeated as a product $(p \cdot n)$ for all $n \geq 0, n \in (r_n)$. Because of this rule in the prime factorization of every single $n!$ as product of prime number with the corresponding exponents in the complete given numerical sequence of factorials $(r_{n!}) = (0!, 1!, 2!, \ldots, n!)$, where one finds the corresponding numerical sequence of exponents $ord_p\,(r_{n!}) = ord_p\,(0!, 1!, 2!, \ldots, n!)$, the outcome of this is a change of exponent only for those numbers from factorials (or members of the numerical sequence of factorials) with the values equal to $(p \cdot n)!$. That means that for one given prime number p all numbers of factorials (or members of the numerical sequence of factorials $(p \cdot n)!$ and $(p \cdot n + 0, 1, 2, \ldots, (p-1))!$) have got p-times the same exponent $ord_p\,(p \cdot n)! = ord_p\,(p \cdot n + 0, 1, 2, \ldots, (p-1))!$.

Those are all the rules for the numerical sequence $\left(A_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n).

Below is shown how one finds the complete numerical sequence of exponents which corresponds to the numerical sequence of factorials $(r_{n!}) = (0!, 1!, 2!, \ldots, (p^\eta - 1)!)$ for a given prime number p from a certain numerical sequence $(r_n) = (0, 1, 2, \ldots, (p^\eta - 1))$.

Note 2. *Variation I is used to show more clearly the method of how to find the complete numerical sequence of exponents. But in order to find the complete numerical sequence of exponents practically one uses Variation II.c.*

2.3. **The formula which shows how one finds the complete numerical sequence of exponents which corresponds to the numerical sequence of factorials $(r_{n!}) = (0!, 1!, 2!, ..., (p^\eta - 1)!)$ for a given prime number p from a certain numerical sequence $(r_n) = (0, 1, 2, ..., (p^\eta - 1))$.** Variation I.

With Formula (2.12) again:

$$\begin{aligned}
&(\overset{\triangleq p^\eta}{A}_{p,(r_n)}) = ((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\ldots,}{a}) + \ldots + (\overset{\eta}{a})) = \\
&= ((0, 1, 2, ..., \overset{1,}{(p-1)}) + (0, 1, 2, ..., \overset{2,\ldots,}{(p-1)}) + \ldots + (0, 1, 2, ..., \overset{\eta}{(p-1)})).
\end{aligned}$$

Then Formula (2.13) again:

$$\begin{aligned}
ord_p\,(r_{n!}) &= ord_p\,(0!, 1!, 2!, ..., (p^\eta - 1)!) = \\
&= \frac{1}{p-1} \cdot ((r_n) - (A_{p,r_n})) = \\
&= \frac{1}{p-1} \cdot \left((0, 1, 2, ..., (p^\eta - 1)) - \left(A_{p,(r_n)}\right)\right).
\end{aligned}$$

Additionally, one takes Remark 1 on page 20 into account under the condition that
$n = 0, 1, 2, ..., (p^\eta - 1)\,, (r_n) = (0, 1, 2, ..., (p^\eta - 1))\,,$
$(r_{n!}) = (0!, 1!, 2!, ..., (p^\eta - 1)!)\,, (a) = (0, 1, 2, ..., (p-1))\,,$
$\eta = 1, 2, 3, ..., \infty, \alpha_{p,p^\eta} = 1, p \geq 2, \mathrm{Nr.}\,(\alpha_{p,0}) = 0, \alpha_{p,0} = 0.$

Variation II.c
At (cf. Formula (2.20), (2.21) and (2.22))

$$((\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + (\underset{2,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + (\underset{3,}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)}) + \underset{...,}{\overset{...}{...}} + (\underset{p}{\overset{\triangleq p^{p^{\eta}}}{A}}_{p,\left(r_{n_1}\right)})) = (\overset{\triangleq p^{p^{\eta+1}}}{A}_{p,\left(r_{n_2}\right)}),$$

$$\begin{aligned}
(\overset{\triangleq p^{p^{\eta}}}{A}\ _{p,\left(r_{n_1}\right)}) &= ((\overset{1,}{a}) + (\overset{2,}{a}) + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}) = \\
&= ((0,1,2,\overset{1,}{...},(p-1)) + (0,1,2,\overset{2,...,}{...},(p-1)) + ... + (0,1,2,\overset{p^{\eta}}{...},(p-1))) = \\
&= (\overset{0,}{0},\overset{1,}{1},\overset{2,...,}{2},...,(p-1),...,\overset{\left(p^{p^{\eta}}-1\right)}{p^{\eta}\cdot(p-1)}),
\end{aligned}$$

$$\begin{aligned}
ord_p\left(r_{(n_2)!}\right) &= ord_p\left(0!,1!,2!,...,\left(p^{p^{\eta+1}}-1\right)!\right) = \frac{1}{p-1}\cdot\left(\left(r_{n_2}\right)-\left(A_{p,\left(r_{n_2}\right)}\right)\right) = \\
&= \frac{1}{p-1}\cdot\left(\left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right)-\left(0,1,2,...,\left(p^{\eta+1}\cdot(p-1)\right)\right)\right).
\end{aligned}$$

Additionally, one takes Remark 1 on page 20 into account under the condition that $0 \leq n_1 \leq \left(p^{p^{\eta}}-1\right)$, $0 \leq n_2 \leq \left(p^{p^{\eta+1}}-1\right)$,
$\left(r_{n_1}\right) = \left(0,1,2,...,\left(p^{p^{\eta}}-1\right)\right)$, $\alpha_{p,\left(p^{p^{\eta}}-1\right)} = p^{\eta}\cdot(p-1)$,
$Nr.\left(\alpha_{p,\left(p^{p^{\eta}}-1\right)}\right) = p^{p^{\eta}}-1$, $\left(r_{n_2}\right) = \left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right)$,
$\alpha_{p,\left(p^{p^{\eta+1}}-1\right)} = p^{\eta+1}\cdot(p-1)$, $Nr.\left(\alpha_{p,\left(p^{p^{\eta+1}}-1\right)}\right) = \left(p^{p^{\eta+1}}-1\right)$,
$(a) = (0,1,2,...,(p-1))$, $0 \leq \eta \leq \infty$, $p \geq 2$,
$\alpha_{p,p^{\eta}} = 1$, $\mathrm{Nr}\left(\alpha_{p,0}\right) = 0$, $\alpha_{p,0} = 0$.

The programme 'A.py' to calculate $\left(A_{p,(r_n)}\right)$ (Variation II.c) can be found in Appendix B on page 49 to be used with the programme 'NumSequence.py' (Helper module) in Appendix A on page 47. One can compare the results to those from Chebyshev's Formula (1.2) - 'Chebyshev.py' - in Appendix D on page 52.

3. THE FORMULA FOR SHORTENED RESULTS FOR THE COMPLETE NUMERICAL SEQUENCE OF EXPONENTS, WHERE ONE USES THE NUMERICAL SEQUENCE $\left(A_{p,(r_{pn})}\right)$ FOR THE GIVEN PRIME NUMBER p AND THE GIVEN NUMERICAL SEQUENCE (r_{pn})

If instead of the numerical sequence $\left(A_{p,(r_n)}\right)$, where $(r_n) = (0,1,2,...,(p^{\eta}-1))$, one uses the numerical sequence $\left(A_{p,(r_{pn})}\right)$, where $(r_{pn}) = ((0\cdot p),(1\cdot p),(2\cdot p),...,((p^{\eta}-1)\cdot p))$ then one can shorten the calculations p-times. The exponent obtained this way in the numerical sequence of exponents one writes p-times repeated and thus, instead of the numerical sequence $(r_{pn}) = ((0\cdot p),(1\cdot p),(2\cdot p),...,((p^{\eta}-1)\cdot p))$, one obtains $(r'_n) = \left(0,1,2,...,\left(p^{\eta+1}+p-1\right)\right)$. (Remark 2, p. 21).

3.1. **Initially one looks at the numerical sequences $\big(A_{p,(r_{pn})}\big)$ for the given prime number p from a certain numerical sequence (r_{pn}) in the development.**

$\underline{p=2}$,

$$(0)\qquad 2^0 \triangleq \overset{2\cdot(2^0-1).}{(0)},$$

$$(1)\qquad 2^1 \triangleq (\overset{0,}{0},\ \overset{2\cdot(2^1-1).}{1}\ ),$$

$$(2)\qquad 2^2 \triangleq (\overset{0,}{0},\overset{2,}{1},\overset{4,}{1},\ \overset{2\cdot(2^2-1).}{2}\ ),$$

$$(3)\qquad 2^3 \triangleq (\overset{0,}{0},\overset{2,}{1},\overset{4,}{1},\overset{6,}{2},\overset{\dots}{1},2,\overset{\dots,}{2},\ \overset{2\cdot(2^3-1).}{3}\ ),$$

$$(4)\qquad 2^4 \triangleq (\overset{0,}{0},\overset{2}{1},\overset{4,}{1},\overset{6,}{2},\overset{\dots}{1},2,2,3,1,2,2,3,2,3,\overset{\dots,}{3},\ \overset{2\cdot(2^4-1).}{4}\ ),$$

etc.

$\underline{p=3}$,

$$(0)\qquad 3^0 \triangleq (\ \overset{3\cdot(3^0-1).}{0}\ ),$$

$$(1)\qquad 3^1 \triangleq (\overset{0,}{0},\overset{3,}{1},\ \overset{3\cdot(3^1-1).}{2}\ ),$$

$$(2)\qquad 3^2 \triangleq (\overset{0,}{0},\overset{3,}{1},\overset{6,}{2},\overset{\dots}{1},2,3,2,\overset{\dots,}{3},\ \overset{3\cdot(3^2-1).}{4}\ ),$$

$$(3)\qquad 3^3 \triangleq (\overset{0,}{0},\overset{3,}{1},\overset{6,}{2},\overset{\dots}{1},2,3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4,\overset{\dots,}{5},\ \overset{3\cdot(3^3-1).}{6}\ ),$$

etc.

$\underline{p=5}$,

$$(0)\qquad 5^0 \triangleq (\ \overset{5\cdot(5^0-1).}{0}\ ),$$

$$(1)\qquad 5^1 \triangleq (\overset{0,}{0},\overset{5,}{1},\overset{10,}{2},\overset{\dots,}{3},\ \overset{5\cdot(5^1-1).}{4}\ ),$$

$$(2)\qquad 5^2 \triangleq (\overset{0,}{0},\overset{5,}{1},\overset{10,}{2},\overset{15,}{3},\overset{\dots}{4},\ 1,2,3,4,5,\ 2,3,4,5,6,\ 3,4,5,6,7,\ 4,5,6,\overset{\dots,}{7},\ \overset{5\cdot(5^2-1).}{8}\ ),$$

$$\begin{aligned}(3)\qquad 5^3 \triangleq (\ &\overset{5\cdot0,}{0},\overset{5\cdot1,}{1},\overset{5\cdot2,}{2},\overset{5\cdot3,}{3},\overset{\dots}{4},\quad 1,2,3,4,5,\quad 2,3,4,5,6,\quad 3,4,5,6,7,\quad 4,5,6,7,8,\\ &1,2,3,4,5,\quad 2,3,4,5,6,\quad 3,4,5,6,7,\quad 4,5,6,7,8,\quad 5,6,7,8,9,\\ &2,3,4,5,6,\quad 3,4,5,6,7,\quad 4,5,6,7,8,\quad 5,6,7,8,9,\quad 6,7,8,9,10,\\ &3,4,5,6,7,\quad 4,5,6,7,8,\quad 5,6,7,8,9,\quad 6,7,8,9,10,\quad 7,8,9,10,11,\\ &4,5,6,7,8,\quad 5,6,7,8,9,\quad 6,7,8,9,10,\quad 7,8,9,10,11,\quad 8,9,10,\overset{\dots,}{11},\ \overset{5\cdot(5^3-1).}{12}\ ),\end{aligned}$$

etc.

3.2. **Law of formation of the numerical sequence $\big(A_{p,(r_{pn})}\big)$ for the given prime number p from a certain numerical sequence (r_{pn}).**

(1) $\big(A_{p,(r_{pn})}\big)$ is a numerical sequence, in which each member $\alpha_{p,(pn)}$ is a positive integer from 0 up to $(p-1)\cdot\eta$, i.e.:

$$0 \leq \alpha_{p,(pn)} \leq (p-1)\cdot\eta. \tag{3.1}$$

(2) (a) Every single member $\alpha_{p,(pn)}$ of the numerical sequence $\left(A_{p,(r_{pn})}\right)$ corresponds to a certain $0 \leq n \leq (p^{\eta}-1)$, i.e.

$$\left(A_{p,(r_{pn})}\right) = \left(\alpha_{p,(p\cdot 0)}, \alpha_{p,(p\cdot 1)}, \alpha_{p,(p\cdot 2)}, ..., \alpha_{p,(p\cdot(p^{\eta}-1))}\right), \tag{3.2}$$

the number of each member $\alpha_{p,(pn)}$ of the numerical sequence $\left(A_{p,(r_{pn})}\right)$ is the number $p \cdot n$, which can be found as

$$Nr.\left(\alpha_{p,(pn)}\right) = p \cdot n = p \cdot (0, 1, 2, ..., (p^{\eta}-1)), \tag{3.3}$$

where

$$0 \leq \alpha_{p,(pn)} \leq (p-1) \cdot \eta.$$

(b) $\left(A_{p,(r_{pn})}\right)$ is the same numerical sequence of the members $\left(\alpha_{p,(r_{pn})}\right)$, which consists of the single members $\alpha_{p,(pn)}$

$$\left(A_{p,(r_{pn})}\right) = \left(\alpha_{p,(r_{pn})}\right) = \left(\alpha_{p,(p\cdot 0)}, \alpha_{p,(p\cdot 1)}, \alpha_{p,(p\cdot 2)}, ..., \alpha_{p,(p\cdot(p^{\eta}-1))}\right),$$

with

$$\left(Nr.Nr.\left(\alpha_{p,(r_{pn})}\right)\right) = (r_{pn}) = ((p\cdot 0), (p\cdot 1), (p\cdot 2), ..., (p\cdot(p^{\eta}-1))), \tag{3.4}$$

where

$$\begin{aligned}
&0 \leq n \leq (p^{\eta}-1), \\
&(r_n) = (0, 1, 2, ..., (p^{\eta}-1)), \\
&(r_{pn}) = p \cdot (0, 1, 2, ..., (p^{\eta}-1)), \\
&p \geq 2, \\
&0 \leq \eta \leq \infty.
\end{aligned}$$

Here (r_n) the numerical sequence of the members n and (r_{pn}) is the numerical sequence of the members $(p \cdot n)$.

(3) The numerical sequence $\left(A_{p,(r_{pn})}\right)$ corresponds to the monitoring number $(k = p^{\eta})$, i.e.

$$\left(A_{p,(r_{pn})}\right) \overset{\wedge}{=} (k = p^{\eta}). \tag{3.5}$$

(4) The sum of all members $\alpha_{p,(pn)}$ of the numerical sequence $\left(A_{p,(r_{pn})}\right)$ equals

$$\sum_{n=0}^{p^{\eta}-1} \alpha_{p,(p\cdot n)} = \frac{1}{2} \cdot (p-1) \cdot \eta \cdot p^{\eta}. \tag{3.6}$$

(5) The last member $\alpha_{p,(p\cdot p^{\eta}-1)}$ of the numerical sequence $\left(A_{p,(r_{pn})}\right)$ equals

$$\alpha_{p,(p\cdot p^{\eta}-1)} = (p-1) \cdot \eta. \tag{3.7}$$

(6) The quantity $Q_{\left(A_{p,(r_{pn})}\right)}$ of all members $\alpha_{p,(pn)}$ of the numerical sequence $\left(A_{p,(r_{pn})}\right)$ equals

$$Q_{\left(A_{p,(r_{pn})}\right)} = p^{\eta}, \tag{3.8}$$

where

$$\begin{aligned}
&p \geq 2, \\
&0 \leq \eta \leq \infty.
\end{aligned}$$

(7) Variation I. The numerical sequence $\left(A_{p,(r_{pn})}\right)$ one has to see as sum of numerical sequences $(a) = (0, 1, 2, ..., (p-1))$:

$$\begin{aligned}
(\overset{\triangleq p^{\eta}}{A}_{p,(r_n)}) &= (\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,\ldots,}{(a)} + \ldots + \overset{\eta}{(a)}) = \\
&= ((\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \underset{1,}{\overset{\ldots,}{\ldots}}, \overset{(p-1)}{(p-1)}) + (\overset{0,}{0}, \overset{1,}{1}, \underset{2,}{\overset{2,}{2}}, \overset{\ldots,}{\ldots}, \overset{(p-1)}{(p-1)}) + \underset{\ldots,}{\overset{\cdots}{\ldots}} + (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \underset{\eta}{\overset{\ldots,}{\ldots}}, \overset{(p-1)}{(p-1)})) = \\
&= (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{\ldots,}{\ldots}, \overset{(p-1),}{(p-1)}, \overset{\ldots,}{\ldots}, \overset{(p^{\eta}-1)}{(\eta\cdot(p-1))}),
\end{aligned}\tag{3.9}$$

$$\begin{aligned}
(\overset{\triangleq p^{\eta}}{A}_{p,(r_{pn})}) &= (\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,\ldots,}{(a)} + \ldots + \overset{\eta}{(a)}) = \\
&= ((\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,}{2}, \underset{1,}{\overset{\ldots,}{\ldots}}, \overset{(p-1)\cdot p,}{(p-1)}) + (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,}{2}, \underset{2,\ldots,}{\overset{\ldots,}{\ldots}}, \overset{(p-1)\cdot p,\ldots}{(p-1)}) + \ldots \\
&\quad \ldots + (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,}{2}, \underset{\eta}{\overset{\ldots,}{\ldots}}, \overset{(p-1)\cdot p}{(p-1)})) = \\
&= (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,}{2}, \overset{\ldots,}{\ldots}, \overset{(p-1)\cdot p,}{(p-1)}, \overset{\ldots,}{\ldots}, \overset{(p^{\eta}-1)\cdot p}{\eta\cdot(p-1)}),
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
1)\ ord_p\left(r_{(pn)!}\right) &= ord_p\left(p\cdot(0,1,2,...,(p^{\eta}-1))\right)! = \\
&= \frac{1}{p-1}\cdot\left(\left(r_{(pn)}\right) - \left(A_{p,\left(r_{(pn)}\right)}\right)\right),
\end{aligned}\tag{3.11}$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n)!}\right)$, which one has obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $(r'_{n!}) = \left(0!, 1!, 2!, ..., \left(p^{\eta+1}-1\right)!\right)$, i.e. if one takes Remark 2 (p. 21) into account one obtains:

$$2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., \left(p^{\eta+1}-1\right)!\right).\tag{3.12}$$

and

$$\begin{aligned}
&(a) = (0, 1, 2, ..., (p-1)), \\
&(r_n) = (0, 1, 2, ..., (p^{\eta}-1)), \\
&(r_{pn}) = p\cdot(0, 1, 2, ..., (p^{\eta}-1)), \\
&(r'_n) = \left(0, 1, 2, ..., \left(p^{\eta+1}-1\right)\right), \\
&p \geq 2, \\
&0 \leq \eta \leq \infty.
\end{aligned}$$

(8) Variation II.a. The sum of the numerical sequences $\left(A_{p,\left(r_{p\cdot n_i}\right)}\right)$, where $1 \leq i \leq \infty$, results in the complete numerical sequence $\left(A_{p,(r_{pn})}\right)$. If every numerical sequence $\left(A_{p,\left(r_{p\cdot n_i}\right)}\right)$ possesses a corresponding monitoring number $(k_i = p^{x_i})$, then the complete numerical sequence $\left(A_{p,(r_{pn})}\right)$ possesses on

its part a corresponding monitoring number ($k = p^{x_1+x_2+x_3+\dots+x_i}$), which is the given prime number p, consisting of the sums of all exponents corresponding to each monitoring number of each numerical sequence $\left(A_{p,(r_{p\cdot n_i})}\right)$:

$$\begin{aligned}
&((\overset{\triangleq p^{x_1}}{A_{p,(r_{p\cdot n_1})}}) + (\overset{\triangleq p^{x_2}}{A_{p,(r_{p\cdot n_2})}}) + (\overset{\triangleq p^{x_3}\dots}{A_{p,(r_{p\cdot n_3})}}) + \dots + (\overset{\triangleq p^{x_i}}{A_{p,(r_{p\cdot n_i})}}))\\
&= (\overset{\triangleq p^{x_1+x_2+x_3+\dots+x_i}}{A_{p,(r_{p\cdot n})}}).
\end{aligned} \tag{3.13}$$

Where

$$\begin{aligned}
(\overset{\triangleq p^{x_1}}{A_{p,(r_{pn_1})}}) &= ((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + \dots + (\overset{x_1}{a})) =\\
&= ((\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{1,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)})) + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{2\ ,\dots,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1),\dots}{(p-1)}) + \dots\\
&\quad \dots + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{x_1}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)}))),\\[2ex]
(\overset{\triangleq p^{x_2}}{A_{p,(r_{pn_2})}}) &= ((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + \dots + (\overset{x_2}{a})) =\\
&= ((\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{1,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)})) + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{2,\ \dots,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1),\dots}{(p-1)}) + \dots\\
&\quad \dots + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{x_2}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)}))),\\
&\ \ \vdots\\
(\overset{\triangleq p^{x_i}}{A_{p,(r_{pn_i})}}) &= ((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + \dots + (\overset{x_i}{a})) =\\
&= ((\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{1,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)})) + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{2,\ \dots,}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1),\dots}{(p-1)}) + \dots\\
&\quad \dots + (\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \underset{x_i}{\overset{p\cdot 2,\dots,}{2}}, \dots, \overset{p\cdot(p-1)}{(p-1)}))),
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
1)\ ord_p\left(r_{(pn)!}\right) &= ord_p\left(p\cdot\left(0,1,2,\dots,\left(p^{x_1+x_2+x_3+\dots+x_i}-1\right)\right)\right)! =\\
&= \frac{1}{p-1}\cdot\left(\left(r_{(pn)}\right) - \left(A_{p,\left(r_{(pn)}\right)}\right)\right),
\end{aligned} \tag{3.15}$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n)!}\right)$, which one has obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $(r'_{n!}) = \left(0!, 1!, 2!, \dots, \left(p^{x_1+x_2+x_3+\dots+x_i+1}-1\right)!\right)$, i.e. if one takes Remark 2 on page 21 into account one obtains:

$$2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, \dots, \left(p^{x_1+x_2+x_3+\dots+x_i+1}-1\right)!\right). \tag{3.16}$$

and

$$
\begin{aligned}
&1 \le x_i \le \infty,\\
&2 \le p \le \infty,\\
&(a) = (0,1,2,...,(p-1)),\\
&0 \le n \le (p^{x_i}-1),\\
&i = 1,2,3,...\\
&(r_{n_i}) = (0,1,2,...,(p^{x_i}-1)),\\
&(r_{pn_i}) = p\cdot(0,1,2,...,(p^{x_i}-1)),\\
&(r_n) = \left(0,1,2,...,\left(p^{x_1+x_2+x_3+...+x_i}-1\right)\right),\\
&(r_{pn}) = p\cdot\left(0,1,2,...,\left(p^{x_1+x_2+x_3+...+x_i}-1\right)\right),\\
&(r'_{n!}) = \left(0,1,2,...,\left(p^{x_1+x_2+x_3+...+x_i+1}-1\right)\right).
\end{aligned}
$$

Variation II.b. The sum of the same numerical sequence $\left(A_{p,\left(r_{p\cdot n_1}\right)}\right)$ results in the complete numerical sequence $\left(A_{p,(r_{p\cdot n})}\right)$. The quantity of these equal numerical sequences $\left(A_{p,(r_{p\cdot n})}\right)$ is denoted as the quantity y. Each of these equal numerical sequences $\left(A_{p,\left(r_{p\cdot n_1}\right)}\right)$ possesses a monitoring number $(k_1 = p^{x_1})$, and therefore the complete numerical sequence $\left(A_{p,(r_{p\cdot n})}\right)$ possesses a corresponding monitoring number $(k = p^{x_1\cdot y})$. All this is written as:

$$
\Big(\underset{1,}{\overset{\triangleq p^{x_1}}{\big(A_{p,\left(r_{p\cdot n_1}\right)}\big)}} + \underset{2,\ \dots,}{\overset{\triangleq p^{x_1}\ \dots}{\big(A_{p,\left(r_{p\cdot n_1}\right)}\big)}} + \dots + \underset{y}{\overset{\triangleq p^{x_1}}{\big(A_{p,\left(r_{p\cdot n_1}\right)}\big)}}\Big) = \overset{\triangleq p^{x_1\cdot y}}{\big(A_{p,(r_{p\cdot n})}\big)}. \tag{3.17}
$$

where

$$
\begin{aligned}
\overset{\triangleq p^{x_1}}{\big(A_{p,\left(r_{n_1}\right)}\big)} &= = \Big(\overset{1,}{(a)} + \big(\overset{2,}{(a)}\big) + \big(\overset{3,\dots,}{(a)}\big) + \dots + \overset{x_1}{(a)}\Big) =\\
&= \Big(\underset{1,}{\big(\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \overset{p\cdot 2,\dots,}{2, \dots,} \overset{p\cdot(p-1)}{(p-1)}\big)} + \underset{2,\ \dots,}{\big(\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \overset{p\cdot 2,\dots,}{2, \dots,} \overset{p\cdot(p-1),\ \dots}{(p-1)}\ \big)} + \dots\\
&\qquad \dots + \underset{x_1}{\big(\overset{p\cdot 0,}{0}, \overset{p\cdot 1,}{1}, \overset{p\cdot 2,\dots,}{2, \dots,} \overset{p\cdot(p-1)}{(p-1)}\big)}\Big),
\end{aligned} \tag{3.18}
$$

where

$$
\begin{aligned}
1)\ \ ord_p\left(r_{(pn)!}\right) &= ord_p\left(p\cdot\left(0,1,2,...,\left(p^{x_1\cdot y}\right)\right)\right)! =\\
&= \frac{1}{p-1}\cdot\left(\left(r_{(pn)}\right) - \left(A_{p,\left(r_{(pn)}\right)}\right)\right),
\end{aligned} \tag{3.19}
$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n)!}\right)$, which one has obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $(r'_{n!}) = \left(0!,1!,2!,...,\left(p^{x_1\cdot y+1}-1\right)!\right)$, i.e. if one takes Remark 2 (p. 21) into account one obtains:

$$
2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!,1!,2!,...,\left(p^{x_1\cdot y+1}-1\right)!\right) \tag{3.20}
$$

and

$$\begin{aligned}
&1 \leq x_1 \leq \infty, \\
&2 \leq p \leq \infty, \\
&(a) = (0, 1, 2, ..., (p-1)), \\
&0 \leq n \leq (p^{x_1} - 1), \\
&(r_{n_1}) = (0, 1, 2, ..., (p^{x_1} - 1)), \\
&(r_{pn_1}) = p \cdot (0, 1, 2, ..., (p^{x_1} - 1)), \\
&(r_n) = (0, 1, 2, ..., (p^{x_1 \cdot y} - 1)), \\
&(r_{pn}) = p \cdot (0, 1, 2, ..., (p^{x_1 \cdot y} - 1)), \\
&(r'_n) = \left(0, 1, 2, ..., \left(p^{x_1 \cdot y + 1} - 1\right)\right).
\end{aligned}$$

Variation II.c It is more practical to use the following development of the numerical sequence $\left(A_{p,(r_{pn_2})}\right)$ for the given prime number p from a certain numerical sequence $\left(r_{(p \cdot n_2)}\right)$ step by step:

$$\begin{aligned}
&((\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{p\cdot n_1})}}}) + (\underset{2,}{\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{p\cdot n_1})}}}) + (\underset{3,}{\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{p\cdot n_1})}}}) + \underset{...,}{\overset{...}{...}} + (\underset{p}{\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{p\cdot n_1})}}})) = \\
&= (\overset{\triangleq p^{p^{\eta+1}}}{A_{p,(r_{pn_2})}}),
\end{aligned} \tag{3.21}$$

of

$$\begin{aligned}
(\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{n_1})}}) &= ((\overset{1,}{(a)}) + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}) = \\
&= ((\overset{0,}{0}, \overset{1,}{1}, \overset{2,...,}{2}, \underset{1,}{...}, \overset{(p-1)}{(p-1)}) + (\overset{0,}{0}, \overset{1,}{1}, \overset{2,...,}{2}, \underset{2,...,}{...}, \overset{(p-1),...}{(p-1)}) + ... \\
&\quad ... + (\overset{0,}{0}, \overset{1,}{1}, \overset{2,...,}{2}, \underset{p^{\eta}}{...}, \overset{(p-1)}{(p-1)})) = \\
&= (\overset{0,}{0}, \overset{1,}{1}, \overset{2,...,}{2}, ..., \overset{(p-1),...,}{(p-1)}, ..., \overset{(p^{p^{\eta}}-1)}{p^{\eta} \cdot (p-1)}),
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
(\overset{\triangleq p^{p^{\eta}}}{A_{p,(r_{pn_1})}}) &= ((\overset{1,}{(a)}) + \overset{2,}{(a)} + \overset{3,...,}{(a)} + ... + \overset{p^{\eta}}{(a)}) = \\
&= ((\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,...,}{2}, \underset{1,}{...}, \overset{(p-1)\cdot p,}{(p-1)}) + (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,...,}{2}, \underset{2,...,}{...}, \overset{(p-1)\cdot p,...}{(p-1)}) + ... \\
&\quad ... + (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,...,}{2}, \underset{p^{\eta}}{...}, \overset{(p-1)\cdot p,}{(p-1)})) = \\
&= (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,...,}{2}, ..., \overset{(p-1)\cdot p,...,}{(p-1)}, ..., \overset{(p^{p^{\eta}}-1)\cdot p}{p^{\eta} \cdot (p-1)}),
\end{aligned} \tag{3.23}$$

where

$$
\begin{aligned}
\text{1) } ord_p\left(r_{(pn)!}\right) &= ord_p\left(p\cdot\left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right)\right)! = \\
&= \frac{1}{p-1}\cdot\left(\left(r_{(pn)}\right)-\left(A_{p,\left(r_{(pn)}\right)}\right)\right),
\end{aligned}
\tag{3.24}
$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n_2)!}\right)$, which we have obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $\left(r'_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}+1}-1\right)\right)$, i.e. if one takes Remark 2 on page 21 into account one obtains:

$$
\text{2) } ord_p\left(r'_{\left(n_2\right)!}\right)=\left(0!,1!,2!,...,\left(p^{p^{\eta+1}+1}-1\right)!\right),
\tag{3.25}
$$

if

$$
\begin{aligned}
&\left(r_{n_1}\right)=\left(0,1,2,...,\left(p^{p^{\eta}}-1\right)\right), \\
&\left(r_{(p\cdot n_1)}\right)=p\cdot\left(0,1,2,...,\left(p^{p^{\eta}}-1\right)\right), \\
&\left(r'_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}+1}-1\right)\right), \\
&\alpha_{p,p\cdot\left(p^{p^{\eta}}-1\right)}=p^{\eta}\cdot(p-1), \\
&Nr.\left(\alpha_{p,p\cdot\left(p^{p^{\eta}}-1\right)}\right)=p\cdot\left(p^{p^{\eta}}-1\right), \\
&\left(r_{n_2}\right)=\left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right), \\
&\left(r_{(p\cdot n_2)}\right)=p\cdot\left(0,1,2,...,\left(p^{p^{\eta+1}}-1\right)\right), \\
&\alpha_{p,p\cdot\left(p^{p^{\eta+1}}-1\right)}=p^{\eta+1}\cdot(p-1), \\
&(a)=(0,1,2,...,(p-1)), \\
&Nr.\left(\alpha_{p,p\cdot\left(p^{p^{\eta+1}}-1\right)}\right)=p\cdot\left(p^{p^{\eta+1}}-1\right), \\
&0\leq\eta\leq\infty,\ \ p\geq 2, \\
&\alpha_{p,p^{\eta}}=1,\ \ \mathrm{Nr}\left(\alpha_{p,0}\right)=0,\ \ \alpha_{p,0}=0.
\end{aligned}
$$

Additionally, one observes Remarks 1 and 2 on pages 20 and 21.

Note 3. *Variation I is used in order to show more clearly the method of how to find the complete numerical sequence of exponents. However, in order to find the complete numerical sequence of exponents practically Variation II.c is used.*

3.3. **The formula for the numerical sequence $\left(A_{p,\left(r_{(pn)}\right)}\right)$ for the given prime number p from a certain numerical sequence $\left(r_{(pn)}\right)$.** Variation I.

Where (cf. Formula (3.10), (3.11)

$$\begin{aligned}
\overset{\triangleq p^{\eta}}{(A_{p,(r_{pn})})} &= (\overset{1,}{(a)} + \overset{2,}{(a)} + \overset{3,\ldots,}{(a)} + \ldots + \overset{\eta}{(a)}) = \\
&= (\underset{1,}{(\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\ldots,}{2,\ldots}, \overset{(p-1)\cdot p,}{(p-1)})} + \underset{2,\ \ldots,}{(\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\ldots,}{2,\ldots}, \overset{(p-1)\cdot p,\ldots}{(p-1)})} + \ldots \\
&\quad \ldots + \underset{\eta}{(\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\ldots,}{2,\ldots}, \overset{(p-1)\cdot p,}{(p-1)})}) = \\
&= (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\ldots,}{2,\ldots}, \overset{(p-1)\cdot p,\ldots,}{(p-1)}, \ldots, \overset{(p^{\eta}-1)\cdot p}{\eta\cdot(p-1)}),
\end{aligned}$$

$$\begin{aligned}
1)\ ord_p\left(r_{(p\cdot n)!}\right) &= ord_p\Big((0\cdot p)!, (1\cdot p)!, (2\cdot p)!, \ldots, (p\cdot(p^{\eta}-1))!\Big) = \\
&= \frac{1}{p-1}\cdot\Big(\left(r_{(p\cdot n)}\right) - \Big(A_{p,\left(r_{(pn)}\right)}\Big)\Big) = \\
&= \frac{1}{p-1}\cdot\Big(p\cdot(r_n) - \Big(A_{p,\left(r_{(pn)}\right)}\Big)\Big) = \\
&= \frac{1}{p-1}\cdot\left(p\cdot(0,1,2,\ldots,(p^{\eta}-1)) - \left(A_{p,(p\cdot(0,1,2,\ldots(p^{\eta}-1)))}\right)\right),
\end{aligned}$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n)!}\right)$, which one has obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $(r'_{n!}) = \left(0!, 1!, 2!, \ldots, \left(p^{\eta+1}-1\right)!\right)$, i.e. if one takes Remark 2 (p. 21) into account one obtains Formula (3.12):

$$2)\ ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, \ldots, \left(p^{\eta+1}-1\right)!\right).$$

Additionally, one has to take Remark 1 (p. 20) into account under the condition that

$n = 0, 1, 2, \ldots, (p^{\eta}-1)$, $(r_n) = (0, 1, 2, \ldots, (p^{\eta}-1))$, $(r'_n) = \left(0, 1, 2, \ldots, \left(p^{\eta+1}-1\right)\right)$,

$(r_{n!}) = (0!, 1!, 2!, \ldots, (p^{\eta}-1)!)$, $(r'_{n!}) = \left(0!, 1!, 2!, \ldots, \left(p^{\eta+1}-1\right)!\right)$,

$\left(r_{(p\cdot n)}\right) = (\overset{0,}{0}, \overset{1,}{p}, \overset{2,}{(2p)}, \ldots, (\overset{p^{\eta}}{(p^{\eta}-1)\cdot p}))$, $(a) = (0, 1, 2, \ldots, (p-1))$,

$\left(r_{(p\cdot n)!}\right) = (\overset{0,}{0!}, \overset{1,}{p!}, \overset{2,}{(2p)!}, \ldots, (\overset{p^{\eta}}{(p^{\eta}-1)\cdot p})!), \eta \geq 0$,

$\alpha_{p,p} = 1, \alpha_{p,p^2} = 1, \alpha_{p,p^3} = 1, \ldots, \alpha_{p,p^{\eta+1}} = 1, p \geq 2$,

$Nr.\,(\alpha_{p,0}) = 0, \alpha_{p,0} = 0.$

Variation II.c

Where (cf. Formula (3.21), (3.22) and (3.23)

$$(\underset{1,}{\overset{\triangleq p^{p^{\eta}}}{(A_{p,(r_{p\cdot n_1})})}} + \underset{2,}{\overset{\triangleq p^{p^{\eta}}}{(A_{p,(r_{p\cdot n_1})})}} + \underset{3,}{\overset{\triangleq p^{p^{\eta}}}{(A_{p,(r_{p\cdot n_1})})}} + \underset{\ldots,}{\overset{\ldots}{\ldots}} + \underset{p}{\overset{\triangleq p^{p^{\eta}}}{(A_{p,(r_{p\cdot n_1})})}}) = \overset{\triangleq p^{p^{\eta+1}}}{(A_{p,(r_{p\cdot n_2})})},$$

$$
\begin{aligned}
\Big(\overset{\triangleq p^{p^{\eta}}}{A}_{p,\left(r_{n_1}\right)}\Big) &= \big((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + \dots + (\overset{p^{\eta}}{a})\big) = \\
&= \big((\overset{0,}{0}, \overset{1,}{1}, \overset{2,\dots,}{2}, \underset{1,}{\dots}, \overset{(p-1)}{(p-1)})\big) + (\overset{0,}{0}, \overset{1,}{1}, \overset{2,\dots,}{2}, \underset{2,\dots,}{\dots}, \overset{(p-1),\dots}{(p-1)}) + \dots \\
&\dots + (\overset{0,}{0}, \overset{1,}{1}, \overset{2,\dots,}{2}, \underset{p^{\eta}}{\dots}, \overset{(p-1)}{(p-1)}))) = \\
&= (\overset{0,}{0}, \overset{1,}{1}, \overset{2,\dots,}{2}, \dots, \overset{(p-1),\dots,}{(p-1)}, \dots, \overset{(p^{p^{\eta}}-1)}{p^{\eta} \cdot (p-1)}),
\end{aligned}
$$

and

$$
\begin{aligned}
\Big(\overset{\triangleq p^{p^{\eta}}}{A}_{p,\left(r_{pn_1}\right)}\Big) &= \big((\overset{1,}{a}) + (\overset{2,}{a}) + (\overset{3,\dots,}{a}) + \dots + (\overset{p^{\eta}}{a})\big) = \\
&= \big((\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\dots,}{2}, \underset{1,}{\dots}, \overset{(p-1)\cdot p,}{(p-1)})\big) + ((\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\dots,}{2}, \underset{2,\dots,}{\dots}, \overset{(p-1)\cdot p,\dots}{(p-1)})) + \dots \\
&\dots + ((\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\dots,}{2}, \underset{p^{\eta}}{\dots}, \overset{(p-1)\cdot p,}{(p-1)}))) = \\
&= (\overset{0\cdot p,}{0}, \overset{1\cdot p,}{1}, \overset{2\cdot p,\dots,}{2}, \dots, \overset{(p-1)\cdot p,\dots,}{(p-1)}, \dots, \overset{(p^{p^{\eta}}-1)\cdot p}{p^{\eta} \cdot (p-1)}),
\end{aligned}
$$

then (cf. Formula (3.24))

$$
\begin{aligned}
1)\ ord_p\left(r_{(pn)!}\right) &= ord_p\left(p \cdot \left(0,1,2,\dots,\left(p^{p^{\eta+1}}-1\right)\right)\right)! = \\
&= \frac{1}{p-1} \cdot \left(\left(r_{(pn)}\right) - \left(A_{p,\left(r_{(pn)}\right)}\right)\right),
\end{aligned}
$$

if one writes every single exponent $ord_p\left(r_{(p\cdot n_2)!}\right)$, which we have obtained, p-times repeated, one obtains the numerical sequence of exponents for the numerical sequence $\left(r'_{n_2}\right) = \left(0,1,2,\dots,\left(p^{p^{\eta+1}+1}-1\right)\right)$, i.e. if one takes Remark 2 on page 21 into account one obtains Formula (3.25):

$$
2)\ ord_p\left(r'_{(n_2)!}\right) = \left(0!,1!,2!,\dots,\left(p^{p^{\eta+1}+1}-1\right)!\right).
$$

Additionally, one takes Remark 1 on page 20 into account under the condition that

$$
\begin{aligned}
& n_1 = 0,1,2,\dots,\left(p^{p^{\eta}}-1\right),\ n_2 = 0,1,2,\dots,\left(p^{p^{\eta+1}}-1\right), \\
& \alpha_{p,p^{\eta}} = 1,\ \mathrm{Nr}\left(\alpha_{p,0}\right) = 0,\ \alpha_{p,0} = 0,\ p \geq 2,\ 0 \leq \eta \leq \infty, \\
& \left(r_{n_1}\right) = \left(0,1,2,\dots,\left(p^{p^{\eta}}-1\right)\right), \\
& \left(r_{(p\cdot n_1)}\right) = \left((p\cdot 0),(p\cdot 1),(p\cdot 2),\dots,\left(p\cdot\left(p^{p^{\eta}}-1\right)\right)\right), \\
& \left(r_{n_2}\right) = \left(0,1,2,\dots,\left(p^{p^{\eta+1}}-1\right)\right), \\
& \left(r_{(p\cdot n_2)}\right) = \left((p\cdot 0),(p\cdot 1),(p\cdot 2),\dots,\left(p\cdot\left(p^{p^{\eta+1}}-1\right)\right)\right), \\
& \left(r'_{n_2}\right) = \left(0,1,2,\dots,\left(p^{p^{\eta+1}+1}-1\right)\right),
\end{aligned}
$$

$$(a) = (0, 1, 2, ..., (p-1))\,.$$

Example 5. Variation I.

$$\begin{aligned}
&p = 3,\\
&\eta = 2,\\
&(r_n) = \left(0, 1, 2, ..., \left(3^2-1\right)\right),\\
&(r'_{n!}) = \left(0!, 1!, 2!, ..., \left(3^{2+1}-1\right)!\right),\\
&\left(r_{(pn)}\right) = 3\cdot\left(0, 1, 2, ..., \left(3^2-1\right)\right),\\
&\left(r_{(pn)!}\right) = \left((0\cdot 3)!, (1\cdot 3)!, (2\cdot 3)!, ..., \left(3\cdot\left(3^2-1\right)\right)!\right),
\end{aligned}$$

then

$$\begin{aligned}
&ord_3\left((0\cdot 3)!, (1\cdot 3)!, (2\cdot 3)!, ..., \left(3\cdot\left(3^2-1\right)\right)!\right) =\\
&= \frac{1}{3-1}\cdot\left(3\cdot\left(0, 1, 2, ..., \left(3^2-1\right)\right) - \left(A_{3,(3\cdot(0,1,2,...,(3^2-1)))}\right)\right) =\\
&= \frac{1}{2}\cdot\left(\left(0, 3, 6, ..., \left(3\cdot\left(3^2-1\right)\right)\right) - (\overset{0,}{0}, \overset{3,}{1}, \overset{6,}{2}, \overset{9,}{1}, \overset{12,}{2}, \overset{15,}{3}, \overset{18,}{2}, \overset{21,}{3}, \overset{24}{4})\right) =\\
&= (\overset{0,}{0}, \overset{3,}{1}, \overset{6,}{2}, \overset{9,}{4}, \overset{12,}{5}, \overset{15,}{6}, \overset{18,}{8}, \overset{21,}{9}, \overset{24}{10})
\end{aligned}$$

and

$$\begin{aligned}
3^2 &\triangleq (\overset{0,}{0}, \overset{1,}{1}, \overset{2,}{2}, \overset{3,}{1}, \overset{4,}{2}, \overset{5,}{3}, \overset{6,}{2}, \overset{7,}{3}, \overset{8}{4}) =\\
&= \left(A_{3,(0,1,2,...(3^2-1))}\right) \triangleq\\
&\triangleq \left(A_{3,(3\cdot(0,1,2,...(3^2-1)))}\right) =\\
&= (\overset{0,}{0}, \overset{3,}{1}, \overset{6,}{2}, \overset{9,}{1}, \overset{12,}{2}, \overset{15,}{3}, \overset{18,}{2}, \overset{21,}{3}, \overset{24.}{4}).
\end{aligned}$$

Now one writes every exponent three times repeated and obtains

$$\begin{aligned}
&(\overset{0,}{0}, \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{1}, \overset{4,}{1}, \overset{5,}{1}, \overset{6,}{2}, \overset{7,}{2}, \overset{8,}{2}, \overset{9,}{4}, \overset{10,}{4}, \overset{11,}{4}, \overset{12,}{5}, \overset{13,}{5}, \overset{14,}{5}, \overset{15,}{6}, \overset{16,}{6}, \overset{17,}{6},\\
&\ \overset{18,}{8}, \overset{19,}{8}, \overset{20,}{8}, \overset{21,}{9}, \overset{22,}{9}, \overset{23,}{9}, \overset{24,}{10}, (\overset{25,}{10}, \overset{(3^3-1).}{10)}\ ).
\end{aligned}$$

Then this numerical sequence of exponents belongs to

$$\begin{aligned}
&ord_3\left(0!, 1!, 2!, ..., \left(3\cdot\left(3^2-1\right)\right)!\right) =\\
&= (\overset{0,}{0}, \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{1}, \overset{4,}{1}, \overset{5,}{1}, \overset{6,}{2}, \overset{7,}{2}, \overset{8,}{2}, \overset{9,}{4}, \overset{10,}{4}, \overset{11,}{4}, \overset{12,}{5}, \overset{13,}{5}, \overset{14,}{5}, \overset{15,}{6}, \overset{16,}{6}, \overset{17,}{6},\\
&\quad \overset{18,}{8}, \overset{19,}{8}, \overset{20,}{8}, \overset{21,}{9}, \overset{22,}{9}, \overset{23,}{9}, \overset{24.}{10}).
\end{aligned}$$

And more precise:

$$ord_3\left(0!,1!,2!,...,\left(3^{2+1}-1\right)!\right)=$$
$$=(\overset{0,}{0},\overset{1,}{0},\overset{2,}{0},\overset{3,}{1},\overset{4,}{1},\overset{5,}{1},\overset{6,}{2},\overset{7,}{2},\overset{8,}{2},\overset{9,}{4},\overset{10,}{4},\overset{11,}{4},\overset{12,}{5},\overset{13,}{5},\overset{14,}{5},\overset{15,}{6},\overset{16,}{6},\overset{17,}{6},$$
$$\overset{18,}{8},\overset{19,}{8},\overset{20,}{8},\overset{21,}{9},\overset{22,}{9},\overset{23,}{9},\overset{24,}{10},\overset{25,}{10},\overset{\left(3^3-1\right).}{10}).$$

Example 6. Variation II.c

$$\begin{aligned}
&p=3,\eta=1,n_1=0,1,2,...,\left(3^{3^0}-1\right),\\
&n_2=0,1,2,...,\left(3^{3^1}-1\right),\\
&3\cdot n_2=0,3,6,9,...,3\cdot\left(3^{3^1}-1\right),\\
&\left(r_{3,n_1}\right)=\left(0,1,2,...,\left(3^{3^0}-1\right)\right),\\
&\left(r_{3,n_2}\right)=\left(0,1,2,...,\left(3^{3^1}-1\right)\right),\\
&\left(r_{3,(3\cdot n_2)}\right)=\left(0,3,6,9,...,3\cdot\left(3^{3^1}-1\right)\right),\\
&(\overset{\triangleq 3^{3^0}}{A}_{3,\left(r_{n_1}\right)})=(a)=(0,1,2),\\
&\left(r'_{n_2}\right)=\left(0,1,2,...,\left(3^{3^1+1}-1\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
(\overset{\triangleq 3^{3^1}}{A}_{3,\left(r_{n_2}\right)})&=((\underset{1,}{\overset{\triangleq 3^{3^0}}{A}_{3,\left(r_{n_1}\right)}})+(\underset{2,}{\overset{\triangleq 3^{3^0}}{A}_{3,\left(r_{n_1}\right)}})+(\underset{3}{\overset{\triangleq 3^{3^0}}{A}_{3,\left(r_{n_1}\right)}}))=\\
&=((0,\underset{1,}{\overset{\triangleq 3^{3^0}}{1}},2)+(0,\underset{2,}{\overset{\triangleq 3^{3^0}}{1}},2)+(0,\underset{3}{\overset{\triangleq 3^{3^0}}{1}},2))=\\
&=\overset{\triangleq 3^{3^1}}{(\overset{0,}{0},\overset{1,}{1},\overset{2,}{2},\overset{3,}{1},\overset{...}{2},3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4,\overset{...,}{5},\overset{3^{3^1}-1.}{6})}.
\end{aligned}$$

At

$$(\overset{\triangleq 3^{3^1}}{A}_{3,\left(r_{(3\cdot n_2)}\right)})=$$

$$=\overset{\triangleq 3^{3^1}}{(\overset{(3\cdot 0),}{0},\overset{(3\cdot 1),}{1},\overset{(3\cdot 2),}{2},\overset{(3\cdot 3),}{1},\overset{...}{2},3,2,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4,\overset{...,}{5},\overset{\left(3\cdot\left(3^{3^1}-1\right)\right).}{6})}.$$

Then

$$\begin{aligned}
ord_3\left(r_{(3\cdot n_2)!}\right) &= ord_3\left((3\cdot 0)!, (3\cdot 1)!, (3\cdot 2)!, ..., \left(3\cdot\left(3^{3^1}-1\right)\right)!\right) = \\
&= \frac{1}{3-1}\cdot\left(\left(r_{(3\cdot n_2)}\right)-\left(A_{3,\left(r_{(3\cdot n_2)}\right)}\right)\right) = \\
&= \frac{1}{3-1}\cdot\left(3\cdot\left(r_{n_2}\right)-\left(A_{3,\left(r_{n_2}\right)}\right)\right) = \\
&= \frac{1}{3-1}\cdot((0,3,6,9,12,15,...,72,75,78) - \\
&\quad -(0,1,2,1,2,3,2\ ,3,4,1,2,3,2,3,4,3,4,5,2,3,4,3,4,5,4,5,6)) = \\
&= (0,1,2,4,5,6,8,9,10,13,14,15,17,18,19,21,22,23, \\
&\quad 26,27,28,30,31,32,34,35,36).
\end{aligned}$$

And in the end, if one writes each exponent 3 times repeated, one obtains

$$\begin{aligned}
ord_3\left(r'_{n_2}\right) = ord_3\left(0!, 1!, 2!, ..., \left(3^{3^1+1}-1\right)!\right) = \\
= (0,0,0,1,1,1,2,2,2,4,4,4,5,5,5,6,6,6,8,8,8, \\
9,9,9,10,10,10,13,13,13,14,14,14,15,15,15, \\
17,17,17,18,18,18,19,19,19,21,21,21,22,22,22, \\
23,23,23,26,26,26,27,27,27,28,28,28,30,30,30, \\
31,31,31,32,32,32,34,34,34,35,35,35,36,36,36).
\end{aligned}$$

4. THE NUMERICAL SEQUENCES $\left(B_{p,(r_n)}\right)$ FOR THE GIVEN PRIME NUMBER p FROM A CERTAIN NUMERICAL SEQUENCE (r_n).

One considers the numerical sequences $\left(B_{p,(r_n)}\right)$ for the given prime number p from a given numerical sequence $(r_n) = (0,1,2,...,n)$ of the members $0 \le n \le \infty$ in Table 1 (T.1) in the rightmost column. The numerical sequence $\left(B_{p,(r_n)}\right)$ consists of the members $\beta_{p,n}$, which correspond to every certain n for the given prime number p from a certain numerical sequence (r_n). The exponent $ord_p n!$ for the given prime number p for a certain positive integer n in the prime factorization of $n!$ is the sum of all preceding members of the numerical sequence $\left(B_{p,(r_n)}\right)$ and of the last member β_{p,p^η} of the numerical sequence $\left(B_{p,(r_n)}\right)$, which corresponds to the certain positive integer n. Every member $\beta_{p,(r_n)}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ has got a certain number. We will find the numbers of every member $\beta_{p,(r_n)}$, in order to have every certain member $\beta_{p,(r_n)}$ at its determined place in the numerical sequence $\left(B_{p,(r_n)}\right)$. One can also use this method to find all exponents $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right)$ for the given prime number p in the prime factorization of the given numerical sequence of factorials $\left(r_{n!}\right) = (0!, 1!, 2!, ..., n!)$.

4.1. **The numerical sequences $\left(B_{p,(r_n)}\right)$ in the development.** Below the numerical sequences $\left(B_{p,(r_n)}\right)$ for the given prime number p and a given numerical sequence $(r_n) = (0,1,2,...,n)$ of the members $0 \le n \le \infty$ are shown in their development, before the law of formation of the numerical sequence $\left(B_{p,(r_n)}\right)$ is written.

$\underline{p=2}$,

$$
\begin{aligned}
2^0 &\triangleq ((\overset{0,}{0}), \overset{2^0;}{0};)\\
2^1 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2^1;}{1};)\\
2^2 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{1}, \overset{3,}{0}, \overset{2^2;}{2};)\\
2^3 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{1}, \overset{3,}{0}, \overset{\dots}{2}, 0, 1, \overset{\dots,}{0}, \overset{2^3;}{3};)\\
2^4 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{1}, \overset{3,}{0}, \overset{\dots}{2}, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, \overset{\dots,}{0}, \overset{2^4;}{4};)\\
2^5 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{1}, \overset{3,}{0}, \overset{\dots}{2}, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0,\ 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, \overset{\dots,}{0}, \overset{2^5;}{5};)\\
&\quad \textit{etc.}
\end{aligned}
$$

$\underline{p=3}$,

$$
\begin{aligned}
3^0 &\triangleq ((\overset{0,}{0}), \overset{3^0;}{0};)\\
3^1 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{3^1;}{1};)\\
3^2 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{\dots}{1}, 0, 0, 1, 0, \overset{\dots,}{0}, \overset{3^2;}{2};)\\
3^3 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{\dots}{1}, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, \overset{\dots,}{0}, \overset{3^3;}{3};)\\
3^4 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{\dots}{1}, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3,\\
&\qquad 0, 0, 1, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3,\\
&\qquad 0, 0, 1, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3,\\
&\qquad 0, 0, 1, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, 0, 3,\\
&\qquad 0, 0, 1, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, 0, \overset{\dots,}{0}, \overset{3^4;}{4};)\\
&\quad \textit{etc.}
\end{aligned}
$$

$\underline{p=5}$,

$$
\begin{aligned}
5^0 &\triangleq ((\overset{0,}{0}), \overset{5^0;}{0};)\\
5^1 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{0}, \overset{4,}{0}, \overset{5^1;}{1};)\\
5^2 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{0}, \overset{\dots}{0}, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \overset{\dots,}{0}, \overset{5^2;}{2};)\\
5^3 &\triangleq ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{0}, \overset{\dots}{0}, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0,\ 2,\\
&\qquad 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0,\ 2,\\
&\qquad 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0,\ 2,\\
&\qquad 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0,\ 2,
\end{aligned}
$$

$$0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,\overset{\ldots,}{0},\ \overset{5^3;}{3};)$$

etc.

Remark 3. Firstly, since there is only one logical explanation for having two zeros at the beginning instead of one, the term "Definition" is introduced for the first member $\beta_{p,0}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$, i.e. defined are $N\left(\beta_{p,0}\right) = (0)$ and $(\beta_{p,0}) = (0)$.

Secondly, the number of the first member $\beta_{p,0}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ indeed is equal to p^{-1}, however, such a meaning of the number is not desirable but the number should equal 0.

Thirdly, one looks at each member $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ from two perspectives:

1. Every certain member $\beta_{p,n}$ of the numerical sequence$\left(B_{p,(r_n)}\right)$ has got a number of its own and
2. every certain member $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is a positive integer.

If one wants to find an exponent for a certain n for the given prime number p and a given numerical sequence (r_n), it is going to be:

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=1}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor . \end{aligned} \tag{4.1}$$

Proof.

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=0}^{t \le \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x - \sum_{x=0}^{t \le \log_p n} \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot x = \\ &= \left(\left\lfloor \frac{n}{p^0} \right\rfloor \cdot 0 + \sum_{x=1}^{t \le \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x \right) - \\ &\quad - \left(\left\lfloor \frac{n}{p^{t+1}} \right\rfloor \cdot (t-1) + \sum_{x=1}^{t \le \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-1) \right) = \\ &= \sum_{x=1}^{t \le \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x - \sum_{x=1}^{t \le \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-1) = \\ &= \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-(x-1)) = \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor , \end{aligned}$$

because $\lfloor n/p^{t+1} \rfloor = 0$, because of $p^t \leq n$, then $p^{t+1} > n$ for $t \leq \log_p n$ and $\lfloor n/p^x \rfloor \cdot x = 0$, if $x = 0$. ☐

In the sum this formula shows of how many and of which single certain members of the numerical sequence $\left(B_{p,(r_n)}\right)$ for the given prime number p and a given numerical sequence (r_n) the exponent for a certain n consists. If one uses the lower bound $(x = 0)$, one can find out the number of zeros, ones, twos, up to $x \leq \log_p n$ the sum consists of, which results in an exponent for a certain n for the given prime number p. The first zero, however, which is written in parentheses, cannot be seen in the sum because it is defined. But in fact the numerical sequence $\left(B_{p,(r_n)}\right)$ is used in order to find the exponent $ord_p\,(r_{n!}) = ord_p\,(0!, 1!, 2!, ..., (p^\eta)!)$ for the complete numerical sequence (r_n) for the prime number p. All these numerical sequences $\left(B_{p,(r_n)}\right)$ are written for

$$\begin{aligned}
&0 \leq n \leq p^\eta,\\
&(r_n) = (0, 1, 2, ..., p^\eta)\,,\\
&(r_{n!}) = (0!, 1!, 2!, ..., (p^\eta)!)\,,\\
&0 \leq \eta \leq \infty,\\
&p \geq 2.
\end{aligned}$$

Example 7. For all $p = 5$, $n = 129$, $p = 5$, $n = 129$, $(r_n) = (0, ..., 129)$ we have

$$\begin{aligned}
(B_{5,(0,\ldots,129)}) = ((\overset{0,}{0}), \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{0}, \overset{\ldots}{0}, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \overset{\ldots,}{0}, \overset{5^2,}{2},\\
0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2,\\
0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2,\\
0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2,\\
0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \overset{\ldots,}{0}, \overset{5^3,}{3},\\
0, 0, \overset{\ldots,}{0}, \overset{129,}{0}, \overset{130.}{1}).
\end{aligned}$$

$$\begin{aligned}
ord_p 129! &= \sum_{x=0}^{5^5 \leq 129} \left(\left\lfloor \frac{129}{5^x} \right\rfloor - \left\lfloor \frac{129}{5^{x+1}} \right\rfloor \right) \cdot x =\\
&= \left(\left\lfloor \frac{129}{5^0} \right\rfloor - \left\lfloor \frac{129}{5^1} \right\rfloor \right) \cdot 0 + \left(\left\lfloor \frac{129}{5^1} \right\rfloor - \left\lfloor \frac{129}{5^2} \right\rfloor \right) \cdot 1 +\\
&\quad + \left(\left\lfloor \frac{129}{5^2} \right\rfloor - \left\lfloor \frac{129}{5^3} \right\rfloor \right) \cdot 2 + \left(\left\lfloor \frac{129}{5^3} \right\rfloor - \left\lfloor \frac{129}{5^4} \right\rfloor \right) \cdot 3 =\\
&= 104 \cdot 0 + 20 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 = 31.
\end{aligned}$$

One compares this result to Chebyshevs formula (1.2):

$$ord_p 129! = \sum_{x=1}^{5^3 \leq 129} \left\lfloor \frac{129}{5^x} \right\rfloor = \left\lfloor \frac{129}{5^1} \right\rfloor + \left\lfloor \frac{129}{5^2} \right\rfloor + \left\lfloor \frac{129}{5^3} \right\rfloor = 25 + 5 + 1 = 31.$$

One can present the exponent of a certain positive integer n from the numerical sequence (r_n) for the given prime number p as the sum of all members of the numerical sequence $\left(B_{p,(r_n)}\right)$ as follows:

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \sum_{x=0}^{p^x \leq n} \beta_{p,x} = \\ &= \beta_{p,0} + \beta_{p,1} + \beta_{p,2} + ... + \beta_{p,n}. \end{aligned}$$

Or

$$\begin{aligned} & ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right) = \left(\sum_{x=0}^{p^x \leq 0} \left(\left\lfloor \frac{0}{p^x} \right\rfloor - \left\lfloor \frac{0}{p^{x+1}} \right\rfloor \right) \cdot x, \right. \\ & \sum_{x=0}^{p^x \leq 1} \left(\left\lfloor \frac{1}{p^x} \right\rfloor - \left\lfloor \frac{1}{p^{x+1}} \right\rfloor \right) \cdot x, \sum_{x=0}^{p^x \leq 2} \left(\left\lfloor \frac{2}{p^x} \right\rfloor - \left\lfloor \frac{2}{p^{x+1}} \right\rfloor \right) \cdot x, ... \\ & \left. ..., \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x \right) = \\ = & \left(\sum_{x=0}^{0} \beta_{p,r_x}, ..., \sum_{x=0}^{p^x \leq n} \beta_{p,r_x} \right) = \\ = & \left(\beta_{p,0}, \left(\beta_{p,0} + \beta_{p,1}\right), \left(\beta_{p,0} + \beta_{p,1} + \beta_{p,2}\right), ... \right. \\ & \left. ..., \left(\beta_{p,0} + \beta_{p,1} + \beta_{p,2} + ... + \beta_{p,n}\right)\right). \end{aligned}$$

But in order to have a simpler possibility to actually find the exponent $ord_p\left(r_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., n!\right)$ for the given prime number p for the certain numerical sequence (r_n) in the prime factorization for a certain numerical sequence of factorials $\left(r_{n!}\right) = \left(0!, 1!, 2!, ..., n!\right)$ the following is lacking:
At

$$\begin{aligned} & n = 0, 1, 2, ..., p^i, ..., p^\eta, \\ & (r_n) = \left(0, 1, 2, ..., p^i, ..., p^\eta\right), \\ & i = 0, 1, 2, ..., (\eta - 1), \\ & \eta = \log_p n = 0, 1, 2, ..., \infty, \\ & p \geq 2. \end{aligned}$$

Then

I. 1) $(\beta_{p,0}) = (0)$ (Definition),
   2) $\beta_{p,p^x} = x, \; 0 \leq x \leq \eta$,

II. 1) $\beta_{p,(0+1,2,3,...,(p-1))} = 0$,
   2) $\beta_{p,(p^0+2,3,4,...,(p-1))} = 0$ (Exception),
   3) $\beta_{p,(p^{i+1}+1,2,3,...,(p-1))} = 0$,

4) $\beta_{p,(p^\eta+1,2,3,...,(p-1))} = 0.$

III. Then the number Nr. of every member $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ as recurring positive integer i and as nonrecurring positive integer η

1) $(\beta_{p,0} \;=\; 0) \overset{\triangle}{=} Nr.\,(0) = (0)$ (Definition),

2) $(\beta_{p,p^\eta} = \eta) \overset{\triangle}{=} (Nr.\,(i)) =$
$= p^i \cdot \left((1,2,3,...,(p-1)) + \left(p \cdot \left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right),$

3) $(\beta_{p,p^\eta} = \eta) \overset{\triangle}{=} Nr.\,(\eta) = p^\eta.$

Therefore, if the number of every member $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n) as recurring positive integer i and as nonrecurring positive integer η (not to forget that the first member $(\beta_{p,0}) = (0)$ is defined) is known, one can find the sums $\left(\sum_{x=0}^{0} \beta_{p,r_x}, ..., \sum_{x=0}^{n} \beta_{p,r_x}\right)$ of all members of the numerical sequence $\left(B_{p,(r_n)}\right)$:

$$\begin{aligned} ord_p\,(r_{n!}) &= ord_p\,(0!,1!,2!,...,n!) = \left(\sum_{x=0}^{0} \beta_{p,r_x}, ..., \sum_{x=0}^{n} \beta_{p,r_x}\right) = \\ &= (\beta_{p,0}, (\beta_{p,0}+\beta_{p,1}), (\beta_{p,0}+\beta_{p,1}+\beta_{p,2}), ... \\ &\quad ..., (\beta_{p,0}+\beta_{p,1}+\beta_{p,2}+...+\beta_{p,n})). \end{aligned}$$

4.2. **Law of Formation for the numerical sequence $\left(B_{p,(r_n)}\right)$ for the given prime number p from a certain numerical sequence (r_n).**

(1) $\left(B_{p,(r_n)}\right)$ is a numerical sequence in which every member $\beta_{p,n}$ is a positive integer from 0 up to η, (with $0 \le \eta \le \infty$)

$$\left(B_{p,(r_n)}\right) = \left(\beta_{p,(r_n)}\right) = ((\beta_{p,0}), \beta_{p,1}, \beta_{p,2}, ..., \beta_{p,p^\eta}). \tag{4.2}$$

(2) The sum of the members $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is

$$\sum_{n=0}^{p^\eta} \beta_{p,n} = (\beta_{p,0}) + \beta_{p,1} + \beta_{p,2} + ... + \beta_{p,p^\eta} = \frac{p^\eta - 1}{p-1}. \tag{4.3}$$

(3) The last member β_{p,p^η} of the numerical sequence $\left(B_{p,(r_n)}\right)$ equals

$$\beta_{p,p^\eta} = \eta \tag{4.4}$$

for

$$(r_n) = (0,1,2,...,p^\eta)$$

(4) The quantity of members $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ equals

$$Q_{\left(B_{p,(r_n)}\right)} = p^\eta + 1. \tag{4.5}$$

(5) The numerical sequence $\left(B_{p,(r_n)}\right)$ is equivalent to the corresponding monitoring number $(k = p^\eta)$, i.e.

$$\left(B_{p,(r_n)}\right) \overset{\triangle}{=} (k = p^\eta). \tag{4.6}$$

(6) The number (Nr.) of every member $\beta_{p,n}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is defined by the following formulae:

a) The number of the first member $\beta_{p,0}$ is

$$Nr.\left(\beta_{p,0}\right)=Nr.\left(0\right)=\left(0\right)\text{Definition},$$
$$\left(\beta_{p,0}\right)=\left(0\right)\text{Definition}.$$

b) The numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1},\beta_{p,2},\beta_{p,3},...,\beta_{p,(p^{\eta}-1)}\right)$ is

$$\left(Nr.\left(i\right)\right)=p^{i}\cdot\left[\left(1,2,...,\left(p-1\right)\right)+\left(p\cdot\left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right]. \tag{4.7}$$

At the same time each member is a positive integer i, with $i=0,1,2,...,(\eta-1)$.

c) The number of the last member $\beta_{p,(p^{\eta})}$ is

$$Nr.\left(\eta\right)=p^{\eta}.$$

The last member $\beta_{p,(p^{\eta})}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is simultaneously the last positive integer η.

Additionally, one takes Remark 1 on page 20 into account under the condition that $0\le n\le p^{\eta},0\le i\le\left(\eta-1\right),0\le\eta\le\infty,\left(r_n\right)=\left(0,1,2,...,p^{\eta}\right),$ $\left(r_{n!}\right)=\left(0!,1!,2!,...,\left(p^{\eta}\right)!\right),\beta_{p,1}=0,\ \beta_{p,p}=1,\beta_{p,p^2}=2,...,\beta_{p,p^{\eta}}=\eta,p\ge 2,$ $Nr.\left(\beta_{p,0}\right)=\left(0\right)$ Definition and $\left(\beta_{p,0}\right)=\left(0\right)$ Definition.

4.3. **A formula in order to find a complete numerical sequence of exponents, which correspond to a numerical sequence of factorials** $\left(r_{n!}\right)=\left(0!,1!,2!,...,\left(p^{\eta}\right)!\right)$ **for a given prime number** p.

The numerical sequence $\left(B_{p,(r_n)}\right)$ equals (cf. Formula (4.2) and (4.7)):

$$\left(B_{p,(r_n)}\right)=\left(\beta_{p,(r_n)}\right)=\left(\left(\beta_{p,0}\right),\beta_{p,1},\beta_{p,2},...,\beta_{p,p^{\eta}}\right)=$$
$$=\left(\overset{0,}{(0),}\overset{1,...,}{0,...,}\overset{(p-1),}{0,}\overset{p,}{1,}\overset{(p+1),...,}{0,...,}\overset{(2p-1),}{0,}\overset{2p,}{1,}\overset{(2p+1),...,}{0,...,}\overset{\left(p^2-1\right),}{0,}\overset{p^2,}{2,}\overset{\left(p^2+1\right),...,}{0,...,}\overset{(p^{\eta}-1),}{0,}\overset{p^{\eta},}{\eta}\right).$$

The numbers above each member are defined as follows:

a) The number of the first member $\beta_{p,0}$ is

$$Nr.\left(\beta_{p,0}\right)=Nr.\left(0\right)=\left(0\right)\text{Definition},$$
$$\left(\beta_{p,0}\right)=\left(0\right)\text{Definition}.$$

b) The numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1},\beta_{p,2},\beta_{p,3},...,\beta_{p,(p^{\eta}-1)}\right)$ is

$$\left(Nr.\left(i\right)\right)=p^{i}\cdot\left[\left(1,2,...,\left(p-1\right)\right)+\left(p\cdot\left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right].$$

At the same time each member is a positive integer i, with $i=0,1,2,...,(\eta-1)$.

c) The number of the last member $\beta_{p,(p^{\eta})}$ is

$$Nr.\left(\eta\right)=p^{\eta}.$$

The last member $\beta_{p,(p^{\eta})}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ is simultaneously the last positive integer η.

Then

$$\begin{aligned} ord_p\left(r_{n!}\right) &= ord_p\left(0!, 1!, 2!, ..., (p^\eta)!\right) = \\ &= \left(\sum_{x=0}^{0}\beta_{p,x}, \sum_{x=0}^{1}\beta_{p,x}, \sum_{x=0}^{2}\beta_{p,x}, ..., \sum_{x=0}^{n=p^\eta}\beta_{p,x}\right) = \\ &= \left(\left(\beta_{P,0}\right), \left(\beta_{p,0}+\beta_{p,1}\right), \left(\beta_{p,0}+\beta_{p,1}+\beta_{p,2}\right), ... \\ &\quad ..., \left(\beta_{p,0}+\beta_{p,1}+\beta_{p,2}+...+\beta_{p,p^\eta}\right)\right). \end{aligned} \tag{4.8}$$

Additionally, one takes Remark 1 on page 20 into account under the condition that $0 \leq n \leq p^\eta, 0 \leq i \leq (\eta-1), 0 \leq \eta \leq \infty, (r_n) = (0, 1, 2, ..., p^\eta)$,
$(r_{n!}) = (0!, 1!, 2!, ..., (p^\eta)!), \beta_{p,1} = 0,\ \beta_{p,p} = 1, \beta_{p,p^2} = 2, ..., \beta_{p,p^\eta} = \eta, p \geq 2$,
$Nr.\left(\beta_{p,0}\right) = (0)$ Definition and $\left(\beta_{p,0}\right) = (0)$ Definition.

Example 8.

$$\begin{aligned} &p = 3, \\ &\eta = 3, \\ &i = 0, 1, 2, ..., (3-1), \\ &n = 0, 1, 2, ..., 3^3, \\ &(r_n) = \left(0, 1, 2, ..., 3^3\right), \\ &(r_{n!}) = \left(0!, 1!, 2!, ..., \left(3^3\right)!\right), \\ &Nr.\left(\beta_{3,0}\right) = (0)\,\text{Definition}, \\ &\left(\beta_{3,0}\right) = (0)\,\text{Definition}. \end{aligned}$$

Then

$$\begin{aligned} Nr.\left(0\right) &= Nr.\left(\beta_{3,0}\right) = (0)\,\text{Definition}, \\ \left(Nr.\left(0\right)\right) &= 3^0 \cdot \left[(1,2) + \left(3 \cdot \left(0, 1, 2, ..., \left(3^{3-0-1}-1\right)\right)\right)\right] = \\ &= (1, 4, 7, 10, 13, 16, 19, 22, 25, \\ &\quad\ 2, 5, 8, 11, 14, 17, 20, 23, 26), \\ \left(Nr.\left(1\right)\right) &= 3^1 \cdot \left[(1,2) + \left(3 \cdot \left(0, 1, 2, ..., \left(3^{3-1-1}-1\right)\right)\right)\right] = (3, 12, 21, \\ &\qquad 6, 15, 24), \\ \left(Nr.\left(2\right)\right) &= 3^2 \cdot \left[(1,2) + \left(3 \cdot \left(0, 1, 2, ..., \left(3^{3-2-1}-1\right)\right)\right)\right] = (9, 18), \\ Nr.\left(3\right) &= 3^3 = 27, \end{aligned}$$

Then

$$\begin{aligned} \left(B_{3,(0,1,2,...,3^3)}\right) &= \left(\left(\beta_{3,0}\right), \beta_{3,1}, \beta_{3,2}, ..., \beta_{3,3^3}\right) = \\ &= \left(\overset{0,}{(0)}, \overset{1,}{0}, \overset{2,}{0}, \overset{...}{1}, 0, 0, 1, 0, 0,\ 2, 0, 0, 1, 0, 0, 1, 0, 0, 2,\right. \\ &\qquad \left. 0,\ 0,\ 1,\ 0, 0, 1, 0, \overset{...,}{0}, \overset{3^3}{3}\right) \end{aligned}$$

and after that

$$\begin{aligned}
& ord_3\left(0!, 1!, 2!, ..., \left(3^3\right)!\right) = \left(\sum_{n=0}^{0} \beta_{3,n}, ..., \sum_{n=0}^{3^3} \beta_{3,n}\right) = \\
= \quad & \left(\left(\beta_{3,0}\right), \beta_{3,1}, \beta_{3,2}, ..., \beta_{3,3^3}\right) = \\
= \quad & (\overset{0,}{(0)}, (\overset{0,}{0} + \overset{1,}{0}), (\overset{0,}{0} + \overset{1,}{0} + \overset{2,}{0}), (\overset{0,}{0} + \overset{1,}{0} + \overset{2,}{0} + \overset{3,}{1}), (\overset{0,}{0} + \overset{1,}{0} + \overset{2,}{0} + \overset{3,}{1} + \overset{4,}{0}), ... \\
& ..., (\overset{0,}{0} + \overset{1,}{0} + \overset{2,}{0} + \overset{...}{1} + 0 + 0 + 1 + 0 + 0 + 2 + 0 + 0 + 0 + 1 + 0 + 0 + 1 + 0 + 0 + 2 + \\
& \quad + 0 + 0 + 0 + 1 + 0 + 0 + 1 + 0 + \overset{...,}{0} + \overset{3^3}{3})) = \\
= \quad & (\overset{0,}{0}, \overset{1,}{0}, \overset{2,}{0}, \overset{...}{1}, 1, 1, 2, 2, 2, 4, 4, 4, 5, 5, 5, 6, 6, 6, 8, 8, 8, 9, 9, 9, 10, 10, \overset{...,}{10}, \overset{27.}{13}).
\end{aligned}$$

The programme 'B.py' to calculate $\left(B_{p,(r_n)}\right)$ can be found in Appendix C on page 50 to be used with the programme 'NumSequence.py' (Helper module) in Appendix A on page 47. One can compare the results to those from Chebyshev's Formula (1.2) - 'Chebyshev.py' - in Appendix D on page 52.

5. The formula for shortened results for the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ for the given prime number p from a certain numerical sequence $\left(r_{(pn)}\right)$.

It would be very inconvenient to have an addition of zeros. Therefore one can spare the work by repeatedly writing every exponent for the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ of the exponents $ord_p\left(r_{(pn)!}\right) = ord_p\left(0!, p!, (2p)!, ..., \left(p^{\eta+1}\right)!\right)$, i.e. if considering Remark 2 one obtains instead of the numerical sequence $\left(B_{p,\left(r'_{n!}\right)}\right)$ the numerical sequence of exponents $ord_p\left(r'_{n!}\right) = ord_p\left(0!, 1!, 2!, ..., \left(p^{\eta+1} + p - 1\right)!\right)$. Every member $\beta_{p,(r_{pn})}$ of the numerical sequence $\left(B_{p,(r_n)}\right)$ has got a certain number. We will find the corresponding number of every member $\beta_{p,(r_{pn})}$, in order to have every certain member $\beta_{p,(r_{pn})}$ at its determined place in the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$.

5.1. **The numerical sequences $\left(B_{p,\left(r_{(pn)}\right)}\right)$ in the development.**

$\underline{p = 2}$,

$2^{-1} \triangleq (\overset{2^{-1+1}}{0}),$

$2^{0} \triangleq (\overset{0,}{(0)}, \overset{2^{0+1},}{1}),$

$2^{1} \triangleq (\overset{0,}{(0)}, \overset{2,}{1}, \overset{2^{1+1},}{2}),$

$2^{2} \triangleq (\overset{0,}{(0)}, \overset{2,}{1}, \overset{4,}{2}, \overset{6,}{1}, \overset{2^{2+1},}{3}),$

$2^{3} \triangleq (\overset{0,}{(0)}, \overset{2,}{1}, \overset{4,}{2}, \overset{...}{1}, 3, 1, 2, \overset{...,}{1}, \overset{2^{3+1};}{4}),$

etc.

$\underline{p=3}$,

$3^{-1} \triangleq ((\overset{3^{-1+1}}{0})),$
$3^{0} \triangleq (\,(\overset{0,}{0}), \overset{3^{0+1},}{1}\,),$
$3^{1} \triangleq (\,(\overset{0,}{0}), \overset{3,}{1}, \overset{6,}{1}, \overset{3^{1+1},}{2}\,),$
$3^{2} \triangleq (\,(\overset{0,}{0}), \overset{3,}{1}, \overset{6,}{1}, \overset{\dots}{2}, 1, 1, 2, 1, \overset{\dots,}{1}, \overset{3^{2+1},}{3}\,),$
$3^{3} \triangleq ((\overset{0,}{0}), \overset{3,}{1}, \overset{6,}{1}, \overset{\dots}{2}, 1, 1, 2, 1, 1, 3, 1, 1, 2, 1, 1, 2, 1, 1, 3, 1, 1, 2, 1, 1, 2, 1, \overset{\dots,}{1}, \overset{3^{3+1};}{4}),$
etc.

Remark 4. In fact the first member $\beta_{p,r_{(p\cdot 0)}} = \beta_{p,r_{(0)}} = \beta_{p,0}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ equals 0, because $p^{\eta+1} = p^{-1+1=0}$, where $\eta = -1$. This, however, means that $\beta_{p,0}$ has got the number which is equal to 1, because $p^0 = 1$, although the number equal to 0 would be needed. Therefore, it is defined that $(\beta_{p,0}) = (0)$ and $Nr.\,(\beta_{p,0}) = (0)$.

5.2. The law of formation of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ for the given prime number p from a certain numerical sequence $\left(r_{(pn)}\right)$.

(1) $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is a numerical sequence in which every member $\beta_{p,r_{(pn)}}$ is a positive integer from 0 up to $\eta+1$, (where $0 \le \eta \le \infty$)

$$\begin{aligned}(B_{p,(r_{pn})}) \quad &= \quad (\beta_{p,(r_{pn})}) = ((\beta_{p,(p\cdot 0)}), \beta_{p,(p\cdot 1)}, \beta_{p,(p\cdot 2)}, \dots \beta_{p,p^{\eta+1}}) = \\ &= \quad ((\overset{0,}{0}), \overset{p,\dots,}{1, \dots,} \overset{(p^2-1),}{1,} \overset{p^2,}{2,} \overset{(p^2+1),\dots,}{1, \dots,} \overset{(p^{\eta+1}-1),}{1,} \overset{p^{\eta+1};}{(\eta+1)}).\end{aligned} \tag{5.1}$$

(2) The quantity of members $\beta_{p,\left(r_{(pn)}\right)}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ equals

$$Q_{\left(B_{p,\left(r_{(pn)}\right)}\right)} = p^{\eta} + 1. \tag{5.2}$$

(3) The last member $\beta_{p,p^{\eta+1}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ equals

$$\beta_{p,p^{\eta+1}} = \eta + 1 \tag{5.3}$$

for

$$(r_n) = (0, 1, 2, \dots, p^{\eta}),$$
$$\left(r_{(pn)}\right) = p \cdot (0, 1, 2, \dots, p^{\eta}) = \left((0\cdot p), (1\cdot p), (2\cdot p), \dots, p^{\eta+1}\right).$$

(4) The sum of all members $\beta_{p,r_{(pn)}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ equals

$$\sum_{n=0}^{p^{\eta}} \beta_{p,(pn)} = \frac{p^{\eta+1}-1}{p-1}. \tag{5.4}$$

(5) The numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is equivalent to the corresponding monitoring number $(k = p^{\eta})$

$$\left(B_{p,\left(r_{(pn)}\right)}\right) \overset{\wedge}{=} (k = p^{\eta})\,. \tag{5.5}$$

(6) The number (Nr.) of each member $\beta_{p,r_{(pn)}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is defined by the following formulae:

a) The number of the first member $\beta_{p,0}$ is

$$Nr.\,(\beta_{p,0}) = Nr.\,(0) = (0)\,\text{Definition},$$
$$(\beta_{p,0}) = (0)\,\text{Definition}.$$

b) The numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1\cdot p}, \beta_{p,2\cdot p}, \beta_{p,3\cdot p}, ..., \beta_{p,(p^{\eta}-1)\cdot p}\right)$ is

$$(Nr.\,(i+1)) = p^{i+1} \cdot \left[(1,2,...,(p-1)) + \left(p \cdot \left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right]. \tag{5.6}$$

At the same time each member is a positive integer i, with $i = 0,1,2,...,(\eta-1)$.

c) The number of the last member $\beta_{p,p^{\eta+1}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is

$$Nr.\,(\eta+1) = p^{\eta+1}.$$

The last member $\beta_{p,(p^{\eta+1})}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is simultaneously the last positive integer $\eta+1$.

5.3. **A formula in order to find a complete numerical sequence of exponents, which correspond to a numerical sequence of factorials $\left(r_{(pn)!}\right) = \left(0!, p!, (2p)!, ..., \left(p^{\eta+1}\right)!\right)$ for a given prime number p.**

The numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ equals (cf. Formula (5.1) and (5.6)):

$$\begin{aligned}(B_{p,(r_{pn})}) &= (\beta_{p,(r_{pn})}) = ((\beta_{p,(p\cdot 0)}), \beta_{p,(p\cdot 1)}, \beta_{p,(p\cdot 2)}, ...\beta_{p,p^{\eta+1}}) = \\ &= (\overset{0,}{(0)}, \overset{p,...,}{1,...,} \overset{(p^2-1),}{1,} \overset{p^2,}{2,} \overset{(p^2+1),...,}{1,...,} \overset{(p^{\eta+1}-1),}{1,} \overset{p^{\eta+1};}{(\eta+1)}).\end{aligned}$$

The numbers above each member are defined as follows:

a) The number of the first member $\beta_{p,0}$ is

$$Nr.\,(\beta_{p,0}) = Nr.\,(0) = (0)\,\text{Definition},$$
$$(\beta_{p,0}) = (0)\,\text{Definition}.$$

b) The numerical sequence of the numbers for the numerical sequence of the members $\left(\beta_{p,1\cdot p}, \beta_{p,2\cdot p}, \beta_{p,3\cdot p}, ..., \beta_{p,(p^{\eta}-1)\cdot p}\right)$ is

$$(Nr.\,(i+1)) = p^{i+1} \cdot \left[(1,2,...,(p-1)) + \left(p \cdot \left(0,1,2,...,\left(p^{\eta-i-1}-1\right)\right)\right)\right].$$

At the same time each member is a positive integer i, with $i = 0,1,2,...,(\eta-1)$.

c) The number of the last member $\beta_{p,p^{\eta+1}}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is

$$Nr.\,(\eta+1) = p^{\eta+1}.$$

The last member $\beta_{p,(p^{\eta+1})}$ of the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ is simultaneously the last positive integer $\eta+1$.

Then

$$\begin{aligned}
1)\ ord_p\left(r_{(pn)!}\right) &= ord_p\left((p\cdot 0)!,(p\cdot 1)!,(p\cdot 2)!,...,\left(p^{\eta+1}\right)!\right) = \\
&= \left(\sum_{x=0}^{(p\cdot 0)}\beta_{p,(px)},\sum_{x=0}^{(p\cdot 1)}\beta_{p,(px)},\sum_{x=0}^{(p\cdot 2)}\beta_{p,(px)},...,\sum_{x=0}^{(p\cdot n)=p^{\eta+1}}\beta_{p,(px)}\right) = \\
&= \left(\left(\beta_{p,(p\cdot 0)}\right),\left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}\right),\left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}+\beta_{p,(p\cdot 2)}\right),...\right. \\
(5.7)\qquad & \left. ...,\left(\beta_{p,(p\cdot 0)}+\beta_{p,(p\cdot 1)}+\beta_{p,(p\cdot 2)}+...+\beta_{p,(p^{\eta+1})}\right)\right).
\end{aligned}$$

If one writes in the numerical sequence $\left(B_{p,\left(r_{(pn)}\right)}\right)$ of exponents $ord_p\left(r_{(pn)!}\right)=ord_p\left(0!,p!,(2p)!,...,\left(p^{\eta+1}\right)!\right)$ every exponent p-times repeated, i.e. if one takes Remark 2 on page 21 into account, one obtains for the numerical sequence $\left(B_{p,\left(r'_{n!}\right)}\right)$ the numerical sequence of exponents:

$$(5.8)\qquad 2)\ ord_p\left(r'_{n!}\right)=ord_p\left(0!,1!,2!,...,\left(p^{\eta+1}+p-1\right)!\right).$$

Additionally, one takes Remark 1 on page 20 into account under the condition that:

$$\begin{aligned}
& n=0,1,2,...,p^{\eta}, i=0,1,2,...,(\eta-1), \eta=0,1,2,...,\infty, \\
& (r_n)=(0,1,2,...,p^{\eta}), (r'_n)=\left(0,1,2,...,\left(p^{\eta+1}+p-1\right)\right), \\
& (r_{n!})=(0!,1!,2!,...,(p^{\eta})!), (r'_{n!})=\left(0!,1!,2!,...,\left(p^{\eta+1}+p-1\right)!\right), \\
& \left(r_{(p\cdot n)}\right)=\left(0,p,(2p),...,\left(p^{\eta+1}\right)\right), \left(r_{(p\cdot n)!}\right)=\left(0!,p!,(2p)!,...,\left(p^{\eta+1}\right)!\right), \\
& \beta_{p,p}=1, \beta_{p,p^2}=2, \beta_{p,p^3}=3,...,\beta_{p,p^{\eta+1}}=\eta+1, p\geq 2, \\
& Nr.\left(\beta_{p,0}\right)=(0)\ \text{Definition and}\ \left(\beta_{p,0}\right)=(0)\ \text{Definition.}
\end{aligned}$$

Example 9.

$$\begin{aligned}
& n=0,1,2,...,3^3, (r_n)=\left(0,1,2,...,3^3\right), \\
& (r_{n!})=\left(0!,1!,2!,...,\left(3^3\right)!\right), \\
& (r'_n)=\left(0,1,2,...,\left(3^{3+1}+3-1\right)\right), \\
& (r'_{n!})=\left(0!,1!,2!,...,\left(3^3+3-1\right)!\right), \\
& \left(r_{(pn)}\right)=\left(r_{(3n)}\right)=\left((3\cdot 0),(3\cdot 1),(3\cdot 2),...,\left(3\cdot 3^3\right)\right), \\
& \left(r_{(pn)!}\right)=\left(r_{(3n)!}\right)=\left((3\cdot 0)!,(3\cdot 1)!,(3\cdot 2)!,...,\left(3\cdot 3^3\right)!\right), \\
& \eta=3, \\
& i=0,1,2, \\
& p=3.
\end{aligned}$$

Then

$$
\begin{aligned}
3^3 \triangleq ((\overset{(3\cdot 0),}{0}), \overset{(3\cdot 1),}{1}, \overset{(3\cdot 2),}{1}, \overset{\dots}{2},\ & 1,1,2,\ 1,1,3,\ 1,1,2,\\
& 1,1,2,\ 1,1,3,\ 1,1,2,\ 1,1,2,\ 1, \overset{\dots,}{1}, \overset{3^{3+1}.}{4}).
\end{aligned}
$$

$$Nr.\,(0) = (0)\ \text{Definition},$$

$$
\begin{aligned}
(Nr.\,(0+1)) &= 3^{0+1}\cdot\left[(1,2)+\left(3\cdot\left(0,1,2,...,\left(3^{3-0-1}-1\right)\right)\right)\right] = (3,12,21,...,75,\\
&\qquad 6,15,24,...,78),\\
(Nr.\,(1+1)) &= 3^{1+1}\cdot\left[(1,2)+\left(3\cdot\left(0,1,2,...,\left(3^{3-1-1}-1\right)\right)\right)\right] = (9,36,63,\\
&\qquad 18,45,72),\\
(Nr.\,(2+1)) &= 3^{2+1}\cdot\left[(1,2)+\left(3\cdot\left(0,1,2,...,\left(3^{3-2-1}-1\right)\right)\right)\right] = (27,54)\,,\\
Nr.\,(3+1) &= 3^{3+1} = 81.
\end{aligned}
$$

Then

$$
\begin{aligned}
& ord_3\left(r_{(3n)!}\right) = \\
= \quad & ord_3\left((3\cdot 0)!,(3\cdot 1)!,(3\cdot 2)!,...,\left(3\cdot 3^3\right)!\right) = \left(\sum_{n=0}^{0}\beta_{3,(3n)},...,\sum_{n=0}^{3\cdot n=3^{3+1}}\beta_{3,(3n)}\right) = \\
= \quad & ((0)\,,(0+1)\,,(0+1+1)\,,(0+1+1+2)\,,...,(0+1+1+2+...+4)) = \\
= \quad & (\overset{0,}{0}, \overset{3,}{1}, \overset{6,}{2},\ \overset{\dots}{4},5,6,\ 8,9,10,\ 13,14,15,\ 17,18,19,\ 21,22,23,\\
& 26,27,28,\ 30,31,32,\ 34,35,\overset{\dots,}{36},\ \overset{81}{40})\,.
\end{aligned}
$$

Therefore

$$ord_3\left(r'_{n!}\right) = ord_3\left(0!,1!,2!,...,\left(3^{3+1}+3-1\right)!\right) =$$

$$
= (\begin{array}{ccccccccc}
\overset{0,}{0,} & \overset{1,}{0,} & \overset{2,}{0,} & \overset{\dots}{1,} & 1, & 1, & 2, & 2, & 2,\\
4, & 4, & 4, & 5, & 5, & 5, & 6, & 6, & 6,\\
8, & 8, & 8, & 9, & 9, & 9, & 10, & 10, & 10,\\
13, & 13, & 13, & 14, & 14, & 14, & 15, & 15, & 15,\\
17, & 17, & 17, & 18, & 18, & 18, & 19, & 19, & 19,\\
21, & 21, & 21, & 22, & 22, & 22, & 23, & 23, & 23,\\
26, & 26, & 26, & 27, & 27, & 27, & 28, & 28, & 28,\\
30, & 30, & 30, & 31, & 31, & 31, & 32, & 32, & 32,\\
34, & 34, & 34, & 35, & 35, & 35, & 36, & 36, & 36,\\
40, & \overset{\dots,}{40,} & \overset{83.}{40}). & & & & & &
\end{array}
$$

REFERENCES

[1] P. L. Chebyshev, 'Mémoire sur les nombres premiers', (St.-Pétersbourg 1854).

[2] N. Koblitz, 'p-adic Numbers, p-adic Analysis, and Zeta-Functions', (New York/ Heidelberg/ Berlin 1977).

[3] G. Tenenbaum and M. Mendès-France, 'The prime numbers and their distribution' *AMS* 1997.

In the Appendix there are the source codes of the programmes to all Sections. They verify the results of the formulae and are written in ‘Python 3.2’.

All programme files can be downloaded from www.maraev.de. They must be saved in one folder so that ‘A.py’ and ‘B.py’ can use ‘NumSequence.py’ automatically.
After typing the file’s name in the Command Prompt one also gives the parameters for $p = 2$ and $eta = 4$: ‘A.py 2 4’ for the calculation.

Appendix A. Programme ‘NumSequence.py’

Helper module to be used with all four programmes.

```
#! /usr/bin/python

import sys

class NumSequence:

              def __init__ ( self, *arg ):
                            if len ( arg ) == 1:
                                          arg = arg[0]
                                          if isinstance ( arg, NumSequence ):
                                                        self.S = arg.S
                                          elif type ( arg ) is list:
                                                        self.S = arg
                            elif len ( arg ) == 2:
                                          self.S = list ( range ( arg[0], arg[1] ) )
                            else:
                                          print ( "NumSequence: Parameter error" )
                                          sys.exit ( 1 )

              def __repr__ ( self ):
                            return repr ( self.S )

              def __add__ ( self, rhs ):
                            if len ( self.S ) == 0:
                                          return NumSequence ( rhs )
                            if len ( rhs.S ) == 0:
                                          return NumSequence ( self )

                            res = list ()
                            for l in self.S:
                                          for r in rhs.S:
                                                        res.append ( l + r )
                            return NumSequence ( res )

              def __sub__ ( self, rhs ):
                            if len ( self.S ) != len ( rhs.S ):
                                          print ( "NumSequence: Differently sized\
                                                sequence on both sides of ’-’ \
```

```
                                                operation." )
                                          sys.exit ( 1 )

                            res = list ()
                            for idx in range ( len ( self.S ) ):
                                          res.append ( self.S[idx] - rhs.S[idx] )
                            return NumSequence ( res )

              ##
              # Multiply each member of the sequence with @p scalar
              #
              def __rmul__ ( self, scalar ):
                            I = iter ( self.S )
                            R = list ()
                            try:
                                          while True:
                                                        R.append ( scalar * next ( I ) )
                            except StopIteration:
                                          pass
                            return NumSequence ( R )

              def __truediv__ ( self, div ):
                            res = list ()
                            for idx in range ( len ( self.S ) ):
                                          res.append ( self.S[idx] / div )
                            return NumSequence ( res )

              def __floordiv__ ( self, div ):
                            res = list ()
                            for idx in range ( len ( self.S ) ):
                                          res.append ( self.S[idx] // div )
                            return NumSequence ( res )

              ##
              # Create subsequence
              def __getitem__ ( self, key ):
                            if type ( key ) is slice:
                                          R = list ()
                                          if key.step is not None:
                                                        print ( "Slice stepping != 1 is \
                                                               not supported" )
                                                        sys.exit ( 1 )
                                          N = iter ( self.S )
                                          n = next ( N )
                                          # Skip leading subsequence
                                          while True:
                                                        if n  >=  key.start:\
                                                                                      break
```

```
                n = next ( N )

            # Copy subsequence
            try:
                while True:
                    if n >= key.stop:
                    R.append ( n )
                    n = next ( N )
            except StopIteration:
                pass
            return NumSequence ( R )
        elif type ( key ) is int:
            return self.S[key]
        else:
            print ( "getitem: Invalid index" )

    def __setitem__ ( self, key, val ):
        if type ( key ) is slice:
            R = list ()
            for N in key:
                self.S[N] = val
        elif type ( key ) is int:
            self.S[key] = val
        else:
            print ( "getitem: Invalid index" )

    def __len__ ( self ):
        return len ( self.S )
```

Appendix B. Programme 'A.py' to calculate $(A_{p,(r_n)})$ (Variation II.c) (Section 2)

```
#! /usr/bin/python

from NumSequence import NumSequence
import sys

if len ( sys.argv ) == 3:
    p  = int ( sys.argv[1] )
    eta = int ( sys.argv[2] )
else:
    print ( "usage: " + sys.argv[0] + " <prime-number> <eta>" )
    sys.exit ( 1 )

def make_A ( p, eta ):
    A = NumSequence ( list ( range ( p ) ) )
    while eta >= 0:
        eta -= 1
        An = NumSequence ( A )
```

```
                            for cnt in range ( p - 1 ):
                                          An = An + A
                            A = An

              return A


A = make_A ( p, eta )
print ( "A=", end="" )
for a in A:
              print ( a, end=", " )
print ()
print ( "p=" + str ( p ) + ", eta=" + str ( eta ) )
print ( "len(A)=" + str ( len ( A ) ) )


ord = (NumSequence ( 0, p ** p ** (eta + 1) ) - A) // (p - 1)
print ( "ord=" + str ( ord ) )
print ( "len(ord)=" + str ( len ( ord ) ) )


ord_s = (p * NumSequence ( 0, p ** p ** (eta + 1) ) - A) // (p - 1)
print ( "ord_s=" + str ( ord_s ) )
print ( "len(ord_s)=" + str ( len ( ord_s ) ) )
```

Appendix C. Programme ‘B.py’ to calculate $\left(B_{p,(r_n)}\right)$ (Section 4)

```
#! /usr/bin/python

from NumSequence import *
import sys

if len ( sys.argv ) == 3:
              p   = int ( sys.argv[1] )
              eta = int ( sys.argv[2] )
else:
              print ( "usage: " + sys.argv[0] + " <prime-number> <eta>" )
              sys.exit ( 1 )


def make_B ( p, eta ):
              N = p ** eta
              # init B with N times zero
              B = NumSequence ( list ( map ( lambda x: 0, range ( N + 1 ) ) ) )
              for i in range ( 0, eta ):
                            Nr_i = p ** i * (NumSequence ( 1, p ) + p * \
                                    NumSequence ( 0, p ** (eta - i - 1)))
#                            print ( "i=" + str ( i ) )
#                            print ( Nr_i )
```

```
                            for idx in Nr_i:
                                          B[idx] = i
              B[N] = eta

              return B



def make_Bs ( p, eta ):
              N = p ** eta
              # init B with N times zero
              B = NumSequence ( list ( map ( lambda x: 0, range ( int ( N/p ) + 1 ) ) ) )
              for i in range ( 1, eta ):
                            Nr_i = p ** i * (NumSequence ( 1, p ) + p * \
                                    NumSequence ( 0, p ** (eta - i - 1)))
#                            print ( "i=" + str ( i ) )
#                            print ( Nr_i )
                            for idx in Nr_i:
                                          B[int ( idx/p )] = i
              B[int ( N/p )] = eta

              return B


B = make_B ( p, eta )
print ( "B=" + str ( B ) )
print ( "len(B)=" + str ( len ( B ) ) )

Bs = make_Bs ( p, eta )
print ( "Bs=" + str ( Bs ) )
print ( "len(Bs)=" + str ( len ( Bs ) ) )

#import os
#os.system ( "ps lax | egrep ’ID|/B’ | grep -v grep" )


ord = list ()
S = 0
for b in B:
              S += b
              ord.append ( S )

print ( "ord=  " + str ( ord ) )
print ( "len(ord)=" + str ( len ( ord ) ) )



ord_s = list ()
```

```
S = 0
for b in Bs:
              S += b
              for idx in range ( p ):
                            ord_s.append ( S )

print ( "ord_s=" + str ( ord_s ) )
print ( "len(ord_s)=" + str ( len ( ord_s ) ) )
```

Appendix D. Programme 'Chebyshev.py' to be used to compare the results of 'A.py' and 'B.py' with the results from Chebyshev's Formula (1.2)

```
#! /usr/bin/python

import sys

if len ( sys.argv ) >= 2:
              p = int ( sys.argv[1] )
              eta = int ( sys.argv[2] )
else:
              print ( "Usage: %s <p> <eta>" % (sys.argv[0],) )
              sys.exit ( 1 )


def chebyshev ( p, n ):
              r = 0
              Z = p
              while Z <= n:
                            r += n // Z
                            Z *= p
              return r


#print ( "ord=", chebyshev ( p, n ) )

N = (p ** eta) - 1

for n in range ( 0, N ):
        print ( chebyshev ( p, n ) )
```

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