DEVELOPMENT OF CHEBYSHEV'S FORMULA FOR PRIME FACTORIZATION OF $n!$ AS A PRODUCT OF THE PRIME NUMBERS p WITH THE CORRESPONDING EXPONENTS, AND THE DERIVATION OF CHEBYSHEV'S FORMULA

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ABSTRACT. One uses Chebyshev's formula or its derivation to obtain the exponent $ord_p n!$ for every given prime number p with the help of the prime factorization of $n!$ for a certain $n = 0, 1, 2, ..., \infty$. In this paper we present the further development on Chebyshev's formula and its derivation. For instance, in Theorem 6.1 for a given n in the form of a difference of products $a_t \cdot p^t$, in Theorem 7.1 for a given n in the form of a certain amount of sums consisting of differences of products $a_t \cdot p^t$ and in Theorem 8.1 for every given n, at once consisting of sums as well as differences of products $a_{t_j} \cdot p^{t_j}$. These theorems offer the possibility to find n by using less summands, resp. minuends of the products $a_{t_j} \cdot p^{t_j}$ as was possible with the derivation of Chebyshev's formula. This only applies if this certain number n can actually be presented with less summands, resp. minuends of products $a_{t_j} \cdot p^{t_j}$.

1. THE FORMULA BY P. L. CHEBYSHEV (1852).

Chebyshev's formula[1] deals with the prime factorization of $n!$ (where $n = 0, 1, 2, \ldots, \infty$) as a product of the prime numbers p with the corresponding exponents.

$$n! = \prod_{p \geq 2}^{p \leq n} p^{\sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor}, \tag{1}$$

or

$$ord_p n! = \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor. \tag{2}$$

2. DERIVATION OF CHEBYSHEV'S FORMULA

One finds a certain exponent of $n!$ for the given prime number p, on using the derivation of Chebyshev's formula.[2]

$$ord_p n! = \frac{n - S_n}{p - 1}, \tag{3}$$

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[1] cf. Tenenbaum, Mendès-France (2000), p.16.

[2] cf. Koblitz (1977), p.7.

Here *Condition 1* applies:

$$\begin{aligned} n &= a_0 + a_1 \cdot p + a_2 \cdot p^2 + ... + a_x \cdot p^x, \\ 0 &\leq a_x \leq p-1, \\ n &= 0, 1, 2, ..., \infty, \\ S_n &= \sum_{x=0}^{p^x \leq n} a_x. \end{aligned}$$

If one compares the derivation of Chebyshev's formula (3) with the formula itself (2) one receives the following:

$$ord_p n! = \frac{n - S_n}{p-1} = \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor,$$

or

$$S_n = n - (p-1) \cdot \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor. \tag{4}$$

Theorem 2.1. *For Condition 1 it is valid that:*

$$\begin{aligned} S_n &= \sum_{x=0}^{p^x \leq n} a_x = \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) = \\ &= n - (p-1) \cdot \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor. \end{aligned} \tag{5}$$

Proof. The sum S_n can be depicted as:

$$\begin{aligned} S_n &= \sum_{x=0}^{p^x \leq n} a_x = \\ &= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) = \\ &= \left(\sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor + \left\lfloor \frac{n}{p^0} \right\rfloor \right) - \\ &- \left(\sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot p + \left\lfloor \frac{n}{p^{t+1}} \right\rfloor \cdot p \right) = \\ &= n + (1-p) \cdot \sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor = \\ &= n - (p-1) \cdot \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor. \end{aligned}$$

Because $\lfloor n/p^{t+1} \rfloor = 0$ because of $p^t \leq n$, then $p^{t+1} > n$ at $t \leq \log_p n$ and $\lfloor n/p^x \rfloor = n$, if $x = 0$

or

$$S_n = \sum_{x=0}^{p^x \leq n} a_x = \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right),$$

then is

$$a_x = \left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p.$$

□

Below my research of Chebyshev's formula and its derivation is presented.

3. Development of Chebyshev's formula

We will prove the following theorem that is reminiscent of Chebyshev's formula (2).

Theorem 3.1. *For* Condition 1 *it is valid that:*

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=1}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor . \end{aligned} \tag{6}$$

Proof. We have

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \right) \cdot x = \\ &= \sum_{x=0}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x - \sum_{x=0}^{t \leq \log_p n} \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot x = \\ &= \left(\sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x + \left\lfloor \frac{n}{p^0} \right\rfloor \cdot 0 \right) - \\ &- \left(\left\lfloor \frac{n}{p^{t+1}} \right\rfloor \cdot (t-1) + \sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-1) \right) = \\ &= \sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x - \sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-1) = \\ &= \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot (x-(x-1)) = \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor \end{aligned}$$

because

$$\sum_{x=0}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x = \left\lfloor \frac{n}{p^0} \right\rfloor \cdot 0 + \sum_{x=1}^{t \leq \log_p n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x = \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor \cdot x$$

and

$$\begin{aligned}\sum_{x=0}^{t\le\log_p n}\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot x &= \sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^{x}}\right\rfloor\cdot(x-1)+\left\lfloor\frac{n}{p^{t+1}}\right\rfloor\cdot t=\\ &= \sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^{x}}\right\rfloor\cdot(x-1)=\\ &= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^{x}}\right\rfloor\cdot(x-1)\end{aligned}$$

because $\lfloor n/p^{t+1}\rfloor = 0$ because of $p^t \le n$, then $p^{t+1} > n$ at $t \le \log_p n$ and $\lfloor n/p^x\rfloor \cdot x = 0$, if $x = 0$.

Therefore

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^{x}}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\right)\cdot x=\\ &= \sum_{x=1}^{p^x\le n}\left(\left\lfloor\frac{n}{p^{x}}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\right)\cdot x=\\ &= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^{x}}\right\rfloor.\end{aligned}$$

□

P.S. The formula from Theorem 3.1 is used to find the exponent of a certain number n for the given prime number p and a given numerical sequence (r_n).[3]

Example 1 (of Theorem 3.1). One considers
$n = 129,\ p = 5,\ (r_n) = (0, ..., 129)$.

$$\begin{aligned}\left(B_{5,(0,\ldots,129)}\right) = & \left(\begin{pmatrix}0,\\0\end{pmatrix}, \overset{1,}{0}, \overset{2,}{0}, \overset{3,}{0}, \overset{\ldots}{0}, 1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,2,\right.\\ & 0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,2,\\ & 0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,2,\\ & 0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,2,\\ & 0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0, \overset{\ldots,}{0}, \overset{5^3,}{3},\\ & \left. 0,0, \overset{\ldots,}{0}, \overset{129.}{0}\right).\end{aligned}$$

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[3] s. Maraev, Said The numerical sequences $\left(A_{p,(r_n)}\right)$ and $\left(B_{p,(r_n)}\right)$ for the given prime number p for a certain numerical sequence (r_n), here: Chapter 2.2: The numerical sequences $\left(B_{p,(r_n)}\right)$ for the given prime number p for a certain numerical sequence (r_n) , in a forthcoming manuscript by S. Maraev.

Then

$$\begin{aligned}
ord_5 129! &= \sum_{x=0}^{5^3 \le n} \left(\left\lfloor \frac{129}{5^x} \right\rfloor - \left\lfloor \frac{129}{5^{x+1}} \right\rfloor \right) \cdot x = \\
&= \left(\left\lfloor \frac{129}{5^0} \right\rfloor - \left\lfloor \frac{129}{5^1} \right\rfloor \right) \cdot 0 + \left(\left\lfloor \frac{129}{5^1} \right\rfloor - \left\lfloor \frac{129}{5^2} \right\rfloor \right) \cdot 1 + \\
&+ \left(\left\lfloor \frac{129}{5^2} \right\rfloor - \left\lfloor \frac{129}{5^3} \right\rfloor \right) \cdot 2 + \left(\left\lfloor \frac{129}{5^3} \right\rfloor - \left\lfloor \frac{129}{5^4} \right\rfloor \right) \cdot 3 = \\
&= 104 \cdot 0 + 20 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 = 31.
\end{aligned}$$

On using Chebyshev's formula (2) one finds:

$$ord_5 129! = \sum_{x=1}^{5^3 \le 129} \left\lfloor \frac{129}{5^x} \right\rfloor = \left\lfloor \frac{129}{5^1} \right\rfloor + \left\lfloor \frac{129}{5^2} \right\rfloor + \left\lfloor \frac{129}{5^3} \right\rfloor = 25 + 5 + 1 = 31.$$

4. Development of the derivation of Chebyshev's formula

Theorem 4.1. *For Condition 2:*

$$\begin{aligned}
n &= a_0 + a_1 \cdot p + a_2 \cdot p^2 + ... + a_x \cdot p^x, \\
n &= 0, 1, 2, ..., \infty, \\
0 &\le a_x \le p - 1, \\
x &= 0, 1, 2, ..., (p^x \le n), \\
a_x &= \left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p, \\
S_n &= \sum_{x=0}^{p^x \le n} a_x = \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right)
\end{aligned}$$

it is valid that

$$ord_p n! = \frac{1}{p-1} \cdot (n - S_n) = \frac{1}{p-1} \cdot \left(n - \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \right). \tag{7}$$

Formula (7) is the same derivation of Chebyshev's formula (3)*.*

Proof. Therefore we have

$$\begin{aligned}
ord_p n! &= \frac{1}{p-1}\cdot\left(n-\sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\right)=\\
&= \frac{1}{p-1}\cdot\left(n-\sum_{x=0}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor+\sum_{x=0}^{p^x\le n}\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)=\\
&= \frac{1}{p-1}\cdot\left(n+\left(-n-\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\right)+\left(\left\lfloor\frac{n}{p^{t+1}}\right\rfloor\cdot p+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p\right)\right)=\\
&= \frac{1}{p-1}\cdot\left(-\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p\right)=\\
&= \frac{1}{p-1}\cdot\left(\sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot(p-1)\right)=\\
&= \frac{1}{p-1}\cdot\left((p-1)\cdot\sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\right)=\sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor.
\end{aligned}$$

Because

$$\sum_{x=0}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor=\left\lfloor\frac{n}{p^0}\right\rfloor+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor=n+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor$$

and

$$\sum_{x=0}^{p^x\le n}\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p=\left\lfloor\frac{n}{p^{t+1}}\right\rfloor\cdot p+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p=\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p$$

because $\lfloor n/p^{t+1}\rfloor = 0$ because of $p^t \le n$, then $p^{t+1} > n$ at $t \le \log_p n$ and $\lfloor n/p^x\rfloor = n$, if $x = 0$. ☐

5. Further development

Theorem 5.1. *For* Condition 2 *it is valid that*

$$\begin{aligned}
ord_p n! &= \sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\cdot\frac{(p^x-1)}{p-1}=\\
&= \sum_{x=1}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\cdot\frac{(p^x-1)}{p-1}=\\
&= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor. \qquad (8)
\end{aligned}$$

Proof of Theorem 5.1. One finds the value of n by using the value of the sum of products $\sum a_x \cdot p^x$.

$$\begin{aligned} n &= a_0 + a_1 \cdot p + a_2 \cdot p^2 + \ldots + a_x \cdot p^x = \\ &= \sum_{x=0}^{p^x \le n} a_x \cdot p^x = \\ &= \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot p^x, \end{aligned}$$

where, according to Theorem 2.1,

$$\begin{aligned} a_x &= \left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p, \\ S_n &= \sum_{x=0}^{x \le \log_p n} a_x = \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right), \\ x &= 0, 1, 2, \ldots, (p^x \le n), \\ n &= 0, 1, 2, \ldots, \infty, \\ 0 &\le a_x \le p - 1. \end{aligned}$$

Then

$$\begin{aligned} ord_p n! &= \frac{1}{p-1} \cdot (n - S_n) = \frac{1}{p-1} \cdot \left(n - \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \right) = \\ &= \frac{1}{p-1} \cdot \left(\sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot p^x - \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \right) = \\ &= \frac{1}{p-1} \cdot \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot (p^x - 1) = \\ &= \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot \frac{(p^x - 1)}{p - 1}. \end{aligned}$$

According to Theorem 4.1 it is

$$\begin{aligned} ord_p n! &= \frac{1}{p-1} \cdot (n - S_n) = \\ &= \frac{1}{p-1} \cdot \left(n - \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \right) = \\ &= \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor, \end{aligned}$$

then

$$
\begin{aligned}
ord_p n! &= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot \frac{(p^x - 1)}{p - 1} = \\
&= \sum_{x=1}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot \frac{(p^x - 1)}{p - 1} = \\
&= \sum_{x=1}^{p^x \leq n} \left\lfloor \frac{n}{p^x} \right\rfloor .
\end{aligned}
$$

□

P.S. One can proof Theorem 5.1 in a different way too:

Theorem 5.2 (Proof of Theorem 5.1, alternative)**.** *Reconsider Formula (*8*) from Theorem* 5.1*.*

Proof. We have

$$
\begin{aligned}
n &= a_0 + a_1 \cdot p + a_2 \cdot p^2 + \ldots + a_x \cdot p^x = \\
&= \sum_{x=0}^{p^x \leq n} a_x \cdot p^x = \\
&= \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot p^x ,
\end{aligned}
$$

where

$$
\begin{aligned}
a_x &= \left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p, \\
S_n &= \sum_{x=0}^{x \leq \log_p n} a_x = \sum_{x=0}^{p^x \leq n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right), \\
x &= 0, 1, 2, \ldots, (p^x \leq n), \\
n &= 0, 1, 2, ..., \infty, \\
0 &\leq a_x \leq p - 1.
\end{aligned}
$$

Then

$$\begin{aligned}
ord_p n! &= \frac{1}{p-1}\cdot(n-S_n) = \\
&= \frac{1}{p-1}\cdot\left(\sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\cdot p^x-\sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\right) = \\
&= \sum_{x=0}^{p^x\le n}\left(\left\lfloor\frac{n}{p^x}\right\rfloor-\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\right)\cdot\frac{(p^x-1)}{p-1} = \\
&= \sum_{x=0}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\frac{(p^x-1)}{p-1}-\sum_{x=0}^{t\le\log_p n}\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\cdot\frac{(p^x-1)}{p-1} = \\
&= \left(\left\lfloor\frac{n}{p^0}\right\rfloor\cdot\frac{\left(p^0-1\right)}{p-1}+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\frac{(p^x-1)}{p-1}\right)- \\
&- \left(\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p\cdot\frac{\left(p^{x-1}-1\right)}{p-1}+\left\lfloor\frac{n}{p^{t+1}}\right\rfloor\cdot p\cdot\frac{(p^t-1)}{p-1}\right) = \\
&= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\left(\frac{p^x-1}{p-1}-p\cdot\frac{\left(p^{x-1}-1\right)}{p-1}\right) = \\
&= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\left(\frac{p^x-1-p^x+p}{p-1}\right) = \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor
\end{aligned}$$

because

$$\begin{aligned}
&\sum_{x=0}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\frac{(p^x-1)}{p-1} = \\
&= \left\lfloor\frac{n}{p^0}\right\rfloor\cdot\frac{\left(p^0-1\right)}{p-1}+\sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\frac{(p^x-1)}{p-1} = \\
&= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot\frac{(p^x-1)}{p-1}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{x=0}^{t\le\log_p n}\left\lfloor\frac{n}{p^{x+1}}\right\rfloor\cdot p\cdot\frac{(p^x-1)}{p-1} = \\
&= \sum_{x=1}^{t\le\log_p n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p\cdot\frac{\left(p^{x-1}-1\right)}{p-1}+\left\lfloor\frac{n}{p^{t+1}}\right\rfloor\cdot p\cdot\frac{(p^t-1)}{p-1} = \\
&= \sum_{x=1}^{p^x\le n}\left\lfloor\frac{n}{p^x}\right\rfloor\cdot p\cdot\frac{\left(p^{x-1}-1\right)}{p-1}
\end{aligned}$$

but $\left\lfloor n/p^{t+1}\right\rfloor\cdot p\cdot(p^t-1)\,/\,(p-1)=0$ because $\left\lfloor n/p^{t+1}\right\rfloor=0$ because of $p^t\le n$, if $p^{t+1}>n$ at $t\le\log_p n$ and $\left\lfloor n/p^0\right\rfloor\cdot\left(p^0-1\right)/\,(p-1)=0$.

Thus,

$$\begin{aligned} ord_p n! &= \sum_{x=0}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot \frac{(p^x - 1)}{p-1} = \\ &= \sum_{x=1}^{p^x \le n} \left(\left\lfloor \frac{n}{p^x} \right\rfloor - \left\lfloor \frac{n}{p^{x+1}} \right\rfloor \cdot p \right) \cdot \frac{(p^x - 1)}{p-1} = \\ &= \sum_{x=1}^{p^x \le n} \left\lfloor \frac{n}{p^x} \right\rfloor . \end{aligned}$$

□

6. The formula for a given n in the form of a difference of products $a_t \cdot p^t$.

Theorem 6.1. *If*

$$\begin{aligned} n &= a_h \cdot p^h - a_{h-1} \cdot p^{h-1} - a_{h-2} \cdot p^{h-2} - \ldots - a_l \cdot p^l, \\ S'_n &= \sum_{t=l}^{h} a_t = a_h - (a_{h-1} + a_{h-2} + \ldots + a_l), \\ l &\le t \le h, \\ 0 &\le a_t \le p-1, \\ l &< ... < h-2 < h-1 < h, \\ 0 &\le l \le \infty, \\ n &= 0, 1, 2, \ldots, \infty. \end{aligned}$$

Then it is valid that

$$ord_p n! = \frac{n - S'_n}{p-1} - (h-l) = \frac{n - \sum_{t=l}^{h} a_t}{p-1} - (h-l). \tag{9}$$

Proof. One has to find the unknown x.

On the one hand (a) $ord_p n!$ is for

$$\begin{aligned} n &= a_h \cdot p^h - a_{h-1} \cdot p^{h-1} - a_{h-2} \cdot p^{h-2} - \ldots - a_l \cdot p^l, \\ S_{n,(1)} &= S'_n = \sum_{t=l}^{h} a_t, \\ (a)\ ord_p n! &= \frac{n - S_{n,(1)}}{p-1} - x = \frac{n - \sum_{t=l}^{h} a_t}{p-1} - x. \end{aligned}$$

On the other hand (b) $ord_p n!$ is determined in the derivation of Chebyshev's formula (3) as

$$\begin{aligned} n &= a_m \cdot p^m + a_{m-1} \cdot p^{m-1} + a_{m-2} \cdot p^{m-2} + \ldots + a_0, \\ S_n &= S_{n,(2)} = \sum_{i=0}^{m} a_i, \\ 0 &\leq i \leq m, \\ p-1 &\geq a_i \geq 0, \\ (b)\ ord_p n! &= \frac{n - S_{n,(2)}}{p-1} = \frac{n - \sum_{i=0}^{m} a_i}{p-1}. \end{aligned}$$

Then one compares the meanings for $ord_p n!$, one has obtained for (a) and (b), in order to find the unknown x:

$$ord_p n! = \frac{n - S_{n,(2)}}{p-1} = \frac{n - S_{n,(1)}}{p-1} - x,$$

therefore

$$x = \frac{n - S_{n,(1)}}{p-1} - \frac{n - S_{n,(2)}}{p-1} = \frac{S_{n,(2)} - S_{n,(1)}}{p-1} = (h-l)$$

or

$$\begin{aligned} S_n = S_{n,(2)} &= S_{n,(1)} + (h-l) \cdot (p-1) = \\ &= \sum_{t=l}^{h} a_t + (h-l) \cdot (p-1)\,, \end{aligned}$$

then

$$\begin{aligned} ord_p n! &= \frac{n - S_n}{p-1} = \frac{n - S_{n,(1)} - (h-l) \cdot (p-1)}{p-1} = \\ &= \frac{n - \sum_{t=l}^{h} a_t - (h-l) \cdot (p-1)}{p-1}. \end{aligned}$$

□

Example 2. For Theorem 6.1:

(1) If using Theorem 6.1: $n = 54080,\ p = 5.$

$$\begin{aligned} 54080 &= 4 \cdot 5^6 - 2 \cdot 5^5 - 3 \cdot 5^4 - 2 \cdot 5^3 - 1 \cdot 5^2 - 4 \cdot 5^1, \\ S_n &= (4 - 2 - 3 - 2 - 1 - 4) + (6 - 1) \cdot (5 - 1) = 12. \end{aligned}$$

(2) If using the derivation of Chebyshev's formula (3): $n = 54080,\ p = 5.$

$$\begin{aligned} 54080 &= 3 \cdot 5^6 + 2 \cdot 5^5 + 1 \cdot 5^4 + 2 \cdot 5^3 + 3 \cdot 5^2 + 1 \cdot 5^1, \\ S_n &= 3 + 2 + 1 + 2 + 3 + 1 = 12, \end{aligned}$$

7. For a given n in the form of a certain amount of sums consisting of differences of products $a_t \cdot p^t$.

Theorem 7.1. *If*

$$\begin{aligned} n &= \left(a_{h_1} \cdot p^{h_1} - a_{h_1-1} \cdot p^{h_1-1} - a_{h_1-2} \cdot p^{h_1-2} - \ldots - a_{l_1} \cdot p^{l_1}\right) + \\ &+ \left(a_{h_2} \cdot p^{h_2} - a_{h_2-1} \cdot p^{h_2-1} - a_{h_2-2} \cdot p^{h_2-2} - \ldots - a_{l_2} \cdot p^{l_2}\right) + \ldots \\ \ldots &+ \left(a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - a_{h_i-2} \cdot p^{h_i-2} - \ldots - a_{l_i} \cdot p^{l_i}\right), \end{aligned}$$

where

$$\begin{aligned}
0 &\leq a_t \leq p-1,\\
h_1 &> h_2 > h_3 > \ldots > h_i,\\
l_1 &> l_2 > l_3 > \ldots > l_i,\\
l_i &< \ldots < h_i - 2 < h_i - 1 < h_i,\\
0 &\leq l_i \leq \infty,\\
i &= 1, 2, 3, \ldots, \infty,\\
n &= 0, 1, 2, \ldots, \infty,\\
n_i &= a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - a_{h_i-2} \cdot p^{h_i-2} - \ldots - a_{l_i} \cdot p^{l_i},\\
n &= \sum_{z=1}^{i} n_z, h = \sum_{z=1}^{i} h_z, l = \sum_{z=1}^{i} l_z,\\
l_i &\leq t \leq h_1.
\end{aligned}$$

Then it is valid that

$$ord_p n! = \frac{n - \sum_{t=l_i}^{h_1} a_t - (h-l) \cdot (p-1)}{p-1}. \tag{10}$$

Proof. On the one hand, if using the derivation of Chebyshev's formula (3), then

$$\begin{aligned}
n &= a_y \cdot p^y + a_{y-1} \cdot p^{y-1} + a_{y-2} \cdot p^{y-2} + \ldots + a_0,\\
S_n &= \sum_{\mu=0}^{y} a_\mu,\\
0 &\leq a_\mu \leq p-1,\\
0 &\leq \mu \leq y,\\
ord_p n! &= \frac{n - S_n}{p-1} = \frac{n - \sum_{\mu=0}^{y} a_\mu}{p-1}.
\end{aligned}$$

On the other hand, under the condition that

$$\begin{aligned}
n &= a_y \cdot p^y + a_{y-1} \cdot p^{y-1} + a_{y-2} \cdot p^{y-2} + \ldots + a_0 =\\
&= \left(a_{h_1} \cdot p^{h_1} - a_{h_1-1} \cdot p^{h_1-1} - a_{h_1-2} \cdot p^{h_1-2} - \ldots - a_{l_1} \cdot p^{l_1}\right) +\\
&+ \left(a_{h_2} \cdot p^{h_2} - a_{h_2-1} \cdot p^{h_2-1} - a_{h_2-2} \cdot p^{h_2-2} - \ldots - a_{l_2} \cdot p^{l_2}\right) + \ldots\\
\ldots &+ \left(a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - a_{h_i-2} \cdot p^{h_i-2} - \ldots - a_{l_i} \cdot p^{l_i}\right) =\\
&= n_1 + n_2 + n_3 + \ldots + n_i
\end{aligned}$$

one finds for every single n_i the expression of the sum of the coefficients S_{n_i} on using Theorem 6.1:

$S_n = S_{n_1} + S_{n_2} + \ldots + S_{n_i}$, $S_{n_1} = (a_{h_1} - a_{h_1-1} - a_{h_1-2} - \ldots - a_{l_1}) + (p-1)\cdot(h_1 - l_1)$, $S_{n_2} = (a_{h_2} - a_{h_2-1} - a_{h_2-2} - \ldots - a_{l_2}) + (p-1)\cdot(h_2 - l_2)$, $\vdots$ $S_{n_i} = (a_{h_i} - a_{h_i-1} - a_{h_i-2} - \ldots - a_{l_i}) + (p-1)\cdot(h_i - l_i)$; then: $S_n = S'_n + \left(\sum_{z=1}^{i} h_z - \sum_{z=1}^{i} l_z\right)\cdot(p-1) = \sum_{t=l_i}^{h_1} a_t + (h-l)\cdot(p-1).$	$S'_n = (S'_{n_1} + S'_{n_2} + \ldots + S'_{n_i})$, $S'_{n_1} = (a_{h_1} - a_{h_1-1} - a_{h_1-2} - \ldots - a_{l_1})$, $S'_{n_2} = (a_{h_2} - a_{h_2-1} - a_{h_2-2} - \ldots - a_{l_2})$, $\vdots$ $S'_{n_i} = (a_{h_i} - a_{h_i-1} - a_{h_i-2} - \ldots - a_{l_i})$; then: $S'_n = \sum_{t=l_i}^{h_1} a_t.$

The following then is the same as what we have obtained in Theorem 6.1:

$$ord_p n! = \frac{n - S_n}{p-1} = \frac{n - S'_n}{p-1} - (h-l) = \frac{n - \sum_{t=l_i}^{h_1} a_t - (h-l)\cdot(p-1)}{p-1}.$$

□

Example 3. For Theorem 7.1.

(1) If using the derivation of Chebyshev's formula (3): $n = 12697806,\ p = 5$.

$$\begin{aligned} 12697806 &= 1\cdot 5^{10} + 1\cdot 5^9 + 2\cdot 5^8 + 2\cdot 5^7 + 2\cdot 5^6 + 3\cdot 5^5 + \\ &\quad +1\cdot 5^4 + 2\cdot 5^3 + 2\cdot 5^2 + 1\cdot 5^1 + 1\cdot 5^0, \\ S_n = \sum a_t &= 1+1+2+2+2+3+1+2+2+1+1 = 18. \end{aligned}$$

(2) if using Theorem 7.1: $n = 12697806,\ p = 5$.

$$\begin{aligned} 12697806 &= \left(2\cdot 5^{10} - 4\cdot 5^9\right) + \left(3\cdot 5^8 - 2\cdot 5^7 - 3\cdot 5^6\right) + \\ &\quad + \left(4\cdot 5^5 - 3\cdot 5^4 - 2\cdot 5^3 - 2\cdot 5^2 - 3\cdot 5^1 - 4\cdot 5^0\right), \\ S_n &= \sum a_t + (h-l)\cdot(p-1) = \\ &= 2-4+3-2-3+4-3-2-2-3-4+ \\ &\quad + ((10-9)+(8-6)+(5-0))\cdot(5-1) = 18. \end{aligned}$$

8. For every given n, at once consisting of sums as well as differences of products $a_{t_j} \cdot p^{t_j}$.

Theorem 8.1. *If*

$$\begin{aligned} n &= a_{\mu_0} \cdot p^{\mu_0} + \left(a_{h_1} \cdot p^{h_1} - a_{h_1-1} \cdot p^{h_1-1} - \ldots - a_{l_1} \cdot p^{l_1}\right) + \\ &+ a_{\mu_1} \cdot p^{\mu_1} + \left(a_{h_2} \cdot p^{h_2} - a_{h_2-1} \cdot p^{h_2-1} - \ldots - a_{l_2} \cdot p^{l_2}\right) + \\ &+ a_{\mu_2} \cdot p^{\mu_2} + \left(a_{h_3} \cdot p^{h_3} - a_{h_3-1} \cdot p^{h_3-1} - \ldots - a_{l_3} \cdot p^{l_3}\right) + \ldots \\ \ldots &+ a_{\mu_{i-1}} \cdot p^{\mu_{i-1}} + \left(a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - \ldots - a_{l_i} \cdot p^{l_i}\right), \end{aligned}$$

it is then valid that

$$ord_p n! = \frac{n - S'_n}{p-1} - (h-l) = \frac{n - \sum_{t=l_i}^{\mu_0} a_t}{p-1} - (h-l)\,, \tag{11}$$

where

$$\begin{aligned} n &= 0, 1, 2, \ldots, \infty, \\ 0 &\leq a_t \leq p-1, \\ l_i &\leq t \leq \mu_0, \\ \mu_{i-1} &> h_i, \\ \mu_0 &> h_1 > l_1 > \mu_1 > h_2 > l_2 > ... > \mu_{i-1} > h_i > l_i, \\ h &= \sum_{z=1}^{i} h_z,\ l = \sum_{z=1}^{i} l_z, \\ m &= \sum_{z=1}^{i} m_z, \\ 0 &\leq l_i \leq \infty, \\ i &= 1, 2, 3, ..., \infty, \\ z &= 1, 2, 3, ..., i. \end{aligned}$$

Proof. One considers n as the sum $n = k + m$ and the given n differently, i.e. one transfers all summands from the products $a_x \cdot p^x$ to the left side of the equation, which are outside the brackets. Then:

$$\begin{aligned} m = n - k &= n - \left(a_{\mu_0} \cdot p^{\mu_0} + a_{\mu_1} \cdot p^{\mu_1} + a_{\mu_2} \cdot p^{\mu_2} + \ldots + a_{\mu_{i-1}} \cdot p^{\mu_{i-1}}\right) = \\ &= \left(a_{h_1} \cdot p^{h_1} - a_{h_1-1} \cdot p^{h_1-1} - \ldots - a_{l_1} \cdot p^{l_1}\right) + \\ &+ \left(a_{h_2} \cdot p^{h_2} - a_{h_2-1} \cdot p^{h_2-1} - \ldots - a_{l_2} \cdot p^{l_2}\right) + \\ &+ \left(a_{h_3} \cdot p^{h_3} - a_{h_3-1} \cdot p^{h_3-1} - \ldots - a_{l_3} \cdot p^{l_3}\right) + \ldots \\ \ldots &+ \left(a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - \ldots - a_{l_i} \cdot p^{l_i}\right). \end{aligned}$$

First one uses Theorem 7.1 for the right side of the equation:

$$\begin{aligned} m &= \left(a_{h_1} \cdot p^{h_1} - a_{h_1-1} \cdot p^{h_1-1} - \ldots - a_{l_1} \cdot p^{l_1}\right) + \\ &+ \left(a_{h_2} \cdot p^{h_2} - a_{h_2-1} \cdot p^{h_2-1} - \ldots - a_{l_2} \cdot p^{l_2}\right) + \\ &+ \left(a_{h_3} \cdot p^{h_3} - a_{h_3-1} \cdot p^{h_3-1} - \ldots - a_{l_3} \cdot p^{l_3}\right) + \ldots \\ \ldots &+ \left(a_{h_i} \cdot p^{h_i} - a_{h_i-1} \cdot p^{h_i-1} - \ldots - a_{l_i} \cdot p^{l_i}\right) = \\ &= m_1 + m_2 + ... + m_i = \sum_{z=1}^{i} m_z, \end{aligned}$$

then

$$ord_p m! = \frac{m - S_m}{p-1} = \frac{(n-k) - \sum_{z=1}^{i} a_{h_z}}{p-1} - (h-l)$$

because

$$\begin{aligned} S_m &= \sum_{z=1}^{i} a_{h_z} - \left(\sum_{z=1}^{i} h_z - \sum_{z=1}^{i} l_z\right) \cdot (p-1) = \\ &= \sum_{z=1}^{i} a_{h_z} - (h-l) \cdot (p-1)\,. \end{aligned}$$

Second, for k the derivation of Chebyshev's formula (3) is used:

$$\begin{aligned} k &= a_{\mu_0} \cdot p^{\mu_0} + a_{\mu_1} \cdot p^{\mu_1} + a_{\mu_2} \cdot p^{\mu_2} + \ldots + a_{\mu_{i-1}} \cdot p^{\mu_{i-1}}, \\ S_k &= \mu_0 + \mu_1 + \mu_2 + \ldots + a_{\mu_{i-1}} = \sum_{z=1}^{i} a_{\mu_{z-1}}, \\ 0 &\leq a_{\mu_{z-1}} \leq p-1 \end{aligned}$$

and

$$ord_p k! = \frac{k - S_k}{p-1} = \frac{k - \sum_{z=1}^{i} a_{\mu_{z-1}}}{p-1}.$$

Then

$$\begin{aligned} ord_p n! &= ord_p k! + ord_p m! = \frac{k - S_k}{p-1} + \frac{(n-k) - S_m}{p-1} = \\ &= \frac{k - \sum_{z=1}^{i} a_{\mu_{z-1}}}{p-1} + \frac{(n-k) - \sum_{z=1}^{i} a_{h_z} - (h-l) \cdot (p-1)}{p-1} = \\ &= \frac{n - S'_n}{p-1} - (h-l) = \frac{n - \sum_{t=l_i}^{\mu_0} a_t}{p-1} - (h-l) \end{aligned}$$

because

$$S'_n = S_k + S_m,$$

or

$$S'_n = \sum_{z=1}^{i} a_{\mu_{z-1}} + \sum_{z=1}^{i} a_{h_z} = \sum_{t=l_i}^{\mu_0} a_t.$$

Therefore

$$ord_p n! = \frac{n - S'_n}{p-1} - (h-l) = \frac{n - \sum_{t=l_i}^{\mu_0} a_t}{p-1} - (h-l)\,.$$

□

Example 4. For Theorem 8.1.

(1) If using the derivation of Chebyshev's formula (3): $n = 26233095,\ p = 5.$

$$\begin{aligned} 26233095 &= 2\cdot 5^{10} + 3\cdot 5^{9} + 2\cdot 5^{8} + 3\cdot 5^{6} + 4\cdot 5^{5} + \\ &\quad +2\cdot 5^{4} + 4\cdot 5^{3} + 3\cdot 5^{2} + 4\cdot 5^{1}, \\ S_n = \sum a_t &= 2+3+2+3+4+2+4+3+4 = 27. \end{aligned}$$

(2) if using Theorem 8.1: $n = 26233095,\ p = 5.$

$$\begin{aligned} 26233095 &= 2\cdot 5^{10} + \left(4\cdot 5^{9} - 3\cdot 5^{8}\right) + \\ &\quad + \left(4\cdot 5^{6} - 2\cdot 5^{4} - 2\cdot 5^{2}\right) + 4\cdot 5^{1}, \\ S_n &= \sum a_t + (h-l)\cdot(p-1) = 2+4-3+4- \\ &\quad -2-2+4+((9-8)+(6-2))\cdot(5-1) = 27. \end{aligned}$$

References

[1] P. L. Chebyshev, 'Mémoire sur les nombres premiers', (St.-Pétersbourg 1854).

[2] N. Koblitz, 'p-adic Numbers, p-adic Analysis, and Zeta-Functions', (New York/ Heidelberg/ Berlin 1977).

[3] G. Tenenbaum and M. Mendès-France, 'The prime numbers and their distribution' *AMS* 1997.

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