

PRIME NUMBERS IN THEIR CORRECT ORDER THE FORMULA FOR THE “SIEVE OF ERATOSTHENES”

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“Until now mathematicians have laboured in vain to detect any order in the series of prime numbers. One is now inclined to believe this to be a secret which the human mind will never penetrate.”
(*Leonhard Euler, Opera omnia, vol. 2, p.241; originally written in French*)

“Today I want to tell you about an area in which I have not done research myself, but which has always fascinated me extraordinarily, and which undoubtedly has fascinated mathematicians from the most ancient prehistory up to our present – namely the question of how the prime numbers are distributed.” (*Don Zagier – Die ersten 50 Millionen Primzahlen (The First 50 Million Prime Numbers), Basel 1977, p.3.*)

(*All translations by the author.*)

ABSTRACT. This paper is based upon the method of the Sieve of Eratosthenes. The first of the formulae in this paper describes this method exactly. The idea of the formula is the following: Let $N \geq 2$ be a given natural number and concerned are the natural numbers n in the interval $R_N = [[2, 2^2), (n \leq N)]$ which do not exceed the value of the number N . If one first subtracts from the interval R_N all multiples of the first prime 2 exceeding 2 and then again adds all natural numbers n , which are missing in the interval R_N in order to be able to subtract all multiples of the second prime 3 exceeding 3, then one can also subtract all multiples of 3 exceeding 3. After that one adds all natural numbers n which are missing in the interval R_N in order to be able to subtract all multiples of the third prime 5 exceeding 5 and can then subtract all multiples of 5 exceeding 5. In this way one proceeds for all primes in their correct order, 7, 11, 13, ... , up to the last prime p_m , which is $\leq \sqrt{N}$, then every natural number n which remains in the interval R_N is a prime itself. If n were a surviving natural number with at least two prime factors then n would be at least as big as N^2 and would therefore lie outside the interval R_N . This way, using this order of primes p_1, \dots, p_m , one can generate a complete list of prime numbers less than p_{m+1}^2 for $m \geq 2$.

After that follow the derivations I and II of this formula.

All three formulae find the prime numbers in their correct order. In the Appendix one finds for each chapter the source code for a programme (using Python 2.7) with the help of which the results can be verified. Up to $N = 10,000,000$ the results are absolutely correct.

1. INTRODUCTION

In many areas of mathematics the method of the Sieve of Eratosthenes¹ is used with algorithms². There has been **no formula exactly to describe the method** of the Sieve of Eratosthenes, named after the Greek mathematician Eratosthenes of Cyrene (276 - 194 BC), in order to identify primes using it. Eratosthenes' idea is quite simple, but very helpful to find the primes in their correct order: Let $N \geq 2$ be a given natural number and focus on the integers n in the interval $R_N = [[2, 2^2), (n \leq N)]$ which do not exceed the value of the number N . If first deleting from the interval R_N all multiples of the first prime 2 exceeding 2, after that all multiples of 3,5,7,... exceeding 3,5,7,... and so on for the multiples of all primes in their correct order up to the last prime p_m , which is $\leq \sqrt{N}$, then every natural number n , not deleted in the interval R_N , is itself prime. If n were a surviving natural number with at least two prime factors, n would be at least as big as N^2 and would therefore lie outside R_N . This way, with the help of this order of primes p_1, \dots, p_m , one can generate a complete list of prime numbers less than p_{m+1}^2 for $m \geq 2$.

For an overview of the problem and its solution one should recall the research concerning the "Sieve of Eratosthenes": The formula by A.-M. Legendre³ (1808), the Moebius function (1831) by A. F. Moebius⁴, the Mangoldt function by Hans von Mangoldt (1895) and the calculation $\pi(x)$ by D. Meissel (1870), even though there is no immediate connection to my formula and its two derivations.

2. THE FORMULA FOR THE "SIEVE OF ERATOSTHENES"

2.1. Description of the procedure in the formula. In the first step $N \geq 2$, a given natural number, is chosen and the natural number n in the interval $R_N = [[2, 2^2), (n \leq N)]$ which do not exceed the value of the number N . If first subtracting from the interval R_N all multiples of the first prime 2 exceeding 2 one obtains a new interval $R_{N,(p_1)} = [[3, 3^2), (n_{(p_1)} \leq N)]$ consisting of the natural numbers $n_{(p_1)} \geq 3$ (which are not divisible by 2) and the first prime $p_1 = 2$, which lies outside this interval $R_{N,(p_1)}$. All natural numbers $n_{(p_1)}$ which lie within the interval $[3, 3^2)$, are the primes p_2, p_3, p_4 in their correct order starting with p_2 .

In step $m \geq 2$ one adds to the result received from step $m - 1$ as interval $R_{N,(p_1, \dots, p_{m-1})} = [[p_m, p_m^2), (n_{(p_1, \dots, p_{m-1})} \leq N)]$ from the natural numbers $n_{(p_1, \dots, p_{m-1})} \geq p_m$ (which are not divisible by p_1, \dots, p_{m-1}) with the primes p_1, \dots, p_{m-1} which lie outside the interval $R_{N,(p_1, \dots, p_{m-1})}$, all missing natural numbers n , after that $n_{(p_1)}$ (which are not divisible by p_1), $n_{(p_1, p_2)}$ (which are not divisible by p_1, p_2), ... up to $n_{(p_1, \dots, p_{m-2})}$ (which are not divisible by p_1, \dots, p_{m-2}) in order to be able to subtract from the interval $R_{N,(p_1, \dots, p_{m-1})}$ all multiples of the prime number p_m exceeding p_m after that. Then one subtracts from this sum all multiples of the prime number p_m exceeding p_m and obtains a new interval $R_{N,(p_1, \dots, p_m)} = [[p_{m+1}, p_{m+1}^2), (n_{(p_1, \dots, p_m)} \leq N)]$ consisting of the natural numbers $n_{(p_1, \dots, p_m)} \geq p_{m+1}$ (which are not divisible by p_1, \dots, p_m) with the primes

¹Information on the sieve of Eratosthenes in e.g.: Hua [3], p.3 and Tenenbaum et al. [5], p.14/15.

²An overview of sieve methods is offered by Halberstam [1].

³Havil [2], p.198-199 and Ore [4], p.69.

⁴An extended discussion of Legendre und Moebius can also be found in Tenenbaum et al. [5], p.15 and p.85.

p_1, \dots, p_m , which lie outside the interval $R_{N,(p_1,\dots,p_m)}$. The prime p_m is $\leq \sqrt{N}$. All natural numbers $n_{(p_1,\dots,p_m)} \geq p_{m+1}$ which lie within the interval $[p_{m+1}, p_{m+1}^2)$ are all the primes in their correct order from p_{m+1} up to $p_{x_m} < p_{m+1}^2$. Here p_{x_m} is the last of the prime numbers.

Exception: The first line of the formula in the term of the addition is deviating from all subsequent lines:

$$(2, 3, 4, \dots, (n \leq p_m - 1)) \cdot p_m.$$

Therefore this line of the formula is the only one to have the interval:

$$R'_N = \begin{cases} \left[2, \left\lfloor \frac{N}{p_m} \right\rfloor \right] & \text{for } \left\lfloor \frac{N}{p_m} \right\rfloor < p_m - 1, \\ \left[2, (p_m - 1) \right] & \text{for } \left\lfloor \frac{N}{p_m} \right\rfloor \geq p_m - 1. \end{cases}$$

The lower bound for a natural number n is greater than or equal to 2. The upper bound for a natural number n is less than or equal to $\lfloor N/p_m \rfloor$ if $\lfloor N/p_m \rfloor$ is less than $p_m - 1$. Or the upper bound for a natural number n is less than or equal to $p_m - 1$ if $\lfloor N/p_m \rfloor$ is greater than or equal to $p_m - 1$.

All subsequent lines of the formula in the term of the addition are written in the same manner:

$$\left\{ \begin{array}{l} R_N = \left(\left\lfloor \frac{p_m}{2} \right\rfloor, \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \\ 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1)} = \left(\left\lfloor \frac{p_m}{3} \right\rfloor, \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \\ 3 \cdot \left(3, 5, 7, \dots, \left(n_{(p_1)} \leq \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1,p_2)} = \left(\left\lfloor \frac{p_m}{5} \right\rfloor, \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \\ 5 \cdot \left(5, 7, 11, \dots, \left(n_{(p_1,p_2)} \leq \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1,p_2,p_3)} = \left(\left\lfloor \frac{p_m}{7} \right\rfloor, \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \\ 7 \cdot \left(7, 11, 13, \dots, \left(n_{(p_1,p_2,p_3)} \leq \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ \dots \\ \\ R_{N,(p_1,\dots,p_{m-2})} = \left(\left\lfloor \frac{p_m}{p_{m-1}} \right\rfloor, \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \\ p_{m-1} \cdot \left(p_{m-1}, p_m, p_{m+1}, \dots, \left(n_{(p_1,\dots,p_{m-2})} \leq \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \right) \cdot p_m. \end{array} \right.$$

All intervals from above can be described as

$$R_N = \left(\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor, \left\lfloor \frac{N}{p_{\nu-1} \cdot p_m} \right\rfloor \right) \text{ for } \nu = 2; m = 2$$

and

$$R_{N,(p_1,\dots,p_{\nu-2})} = \left(\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor, \left\lfloor \frac{N}{p_{\nu-1} \cdot p_m} \right\rfloor \right) \text{ for } \nu = 3, 4, 5, \dots, m; m \geq 3.$$

The interval $R_{N,(p_1,\dots,p_{\nu-2})}$ has to be regarded as

$$\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor < n_{(p_1,\dots,p_{\nu-2})} \leq \left\lfloor \frac{N}{(p_{\nu-1} \cdot p_m)} \right\rfloor.$$

(The explanation for the interval R_N is analogous to the one for the interval $R_{N,(p_1,\dots,p_{\nu-2})}$. Thus it is not explained again here.)

That means:

- (a) if the value $\lfloor p_m/p_{\nu-1} \rfloor$ is not contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, the value of the number next in size is taken which is available in the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.
- (b) if the value $\lfloor p_m/p_{\nu-1} \rfloor$ is contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$ also the value of the number next in size is taken which is available in the complete interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.
- (c) if the value $\lfloor N/(p_{\nu-1} \cdot p_m) \rfloor$ is contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, it remains as such.
- (d) if the value $\lfloor N/(p_{\nu-1} \cdot p_m) \rfloor$ is not contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, the value of the next smaller number is taken which is available in the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.

Remark 1.

- (1) The sum of natural numbers, which belong to the specified interval, is:
 $2, 4, 6, \dots, (2t) + 3, 5, 7, \dots, (2t + 1) = 2, 3, 4, \dots, (2t), (2t + 1)$.
- (2) The difference of natural numbers, which belong to the specified interval, is:
 $2, 3, 4, \dots, (2t), (2t + 1) - 2, 4, 6, \dots, (2t) = 3, 5, 7, \dots, (2t + 1)$.
- (3) The multiplication of natural numbers, which belong to the specified interval, with a natural number $m \geq 2$ is:
 $(1, 2, 3, \dots, n) \cdot m = m, 2m, 3m, \dots, nm$.
- (4) The addition of natural numbers, which belong to the specified interval, with a natural number $m \geq 2$ is:
 $(1, 2, 3, \dots, n) \cdot m = (m, 2m, 3m, \dots, nm) \div m = 1, 2, 3, \dots, n$.

2.2. The formula for the ‘‘Sieve of Eratosthenes’’. First step $m = 1$.

$$\begin{aligned} & {}_{2,R_{N,(p_1)=2},[[3,3^2],(n_{(p_1)} \leq N)]} \\ & 2, (3, 5, 7, 9, \dots, (n_{(p_1)} \leq N)) = \\ & = {}_{R_N=[[2,2^2],(n \leq N)]} \stackrel{R_N=[2,(n \leq \lfloor \frac{N}{2} \rfloor)]}{=} \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right) \right), \end{aligned}$$

step $m \geq 2$.

$$\begin{aligned}
 & 2, 3, 5, \dots, p_m, R_{N, (p_1, \dots, p_m)} = 2, 3, 5, \dots, p_m, \left[[p_{m+1}, p_{m+1}^2], (n_{(p_1, \dots, p_m)} \leq N) \right] \\
 & 2, 3, 5, \dots, p_m, (p_{m+1}, p_{m+2}, p_{m+3}, \dots, p_{x_m}, p_{m+1}^2, \dots, (n_{(p_1, \dots, p_m)} \leq N)) = \\
 & 2, 3, 5, \dots, p_{m-1}, R_{N, (p_1, \dots, p_{m-1})} = 2, 3, 5, \dots, p_{m-1}, \left[[p_m, p_m^2], (n_{(p_1, \dots, p_{m-1})} \leq N) \right] \\
 & = 2, 3, 5, \dots, p_{m-1}, (p_m, p_{m+1}, p_{m+2}, \dots, p_{x_{m-1}}, p_m^2, \dots, (n_{(p_1, \dots, p_{m-1})} \leq N)) + \\
 & + \left\{ \begin{array}{l}
 R'_N = \begin{cases} [2, \lfloor \frac{N}{p_m} \rfloor] \text{ for } \lfloor \frac{N}{p_m} \rfloor < p_{m-1}, \\
 [2, (p_m - 1)] \text{ for } \lfloor \frac{N}{p_m} \rfloor \geq p_{m-1}. \end{cases} \\
 (2, 3, 4, \dots, (n \leq (p_m - 1))) \cdot p_m \\
 \\
 R_N = (\lfloor \frac{p_m}{2} \rfloor, \lfloor \frac{N}{2 \cdot p_m} \rfloor) \\
 2 \cdot (2, 3, 4, \dots, (n \leq \lfloor \frac{N}{2 \cdot p_m} \rfloor)) \cdot p_m \\
 \\
 R_{N, (p_1)} = (\lfloor \frac{p_m}{3} \rfloor, \lfloor \frac{N}{3 \cdot p_m} \rfloor) \\
 3 \cdot (3, 5, 7, \dots, (n_{(p_1)} \leq \lfloor \frac{N}{3 \cdot p_m} \rfloor)) \cdot p_m \\
 \\
 R_{N, (p_1, p_2)} = (\lfloor \frac{p_m}{5} \rfloor, \lfloor \frac{N}{5 \cdot p_m} \rfloor) \\
 5 \cdot (5, 7, 11, \dots, (n_{(p_1, p_2)} \leq \lfloor \frac{N}{5 \cdot p_m} \rfloor)) \cdot p_m \\
 \\
 R_{N, (p_1, p_2, p_3)} = (\lfloor \frac{p_m}{7} \rfloor, \lfloor \frac{N}{7 \cdot p_m} \rfloor) \\
 7 \cdot (7, 11, 13, \dots, (n_{(p_1, p_2, p_3)} \leq \lfloor \frac{N}{7 \cdot p_m} \rfloor)) \cdot p_m \\
 \\
 \dots\dots\dots \\
 R_{N, (p_1, \dots, p_{m-2})} = (\lfloor \frac{p_m}{p_{m-1}} \rfloor, \lfloor \frac{N}{p_{m-1} \cdot p_m} \rfloor) \\
 p_{m-1} \cdot (p_{m-1}, p_m, p_{m+1}, \dots, (n_{(p_1, \dots, p_{m-2})} \leq \lfloor \frac{N}{p_{m-1} \cdot p_m} \rfloor)) \cdot p_m \\
 \\
 R_N = [2, \lfloor \frac{N}{p_m} \rfloor] \\
 - p_m \cdot (2, 3, 4, \dots, (n \leq \lfloor \frac{N}{p_m} \rfloor)) \cdot p_m
 \end{array} \right. \quad -
 \end{aligned}
 \tag{1}$$

The programme for Chapter 2 (“eratosthenes.py”) can be found in the Appendix, page 12.

3. THE FIRST DERIVATION OF THE FORMULA FOR THE “SIEVE OF ERATOSTHENES” IN ITS DEVELOPMENT FROM THE FIRST STEP TO STEP M

3.1. Description of the procedure in the formula. In the first step $N \geq 2$, a given natural number, is chosen and the natural numbers n in the interval $R_N = [[2, 2^2], (n \leq N)]$ which do not exceed the value of the number N . If first subtracting from the interval R_N all multiples of the first prime 2 starting with 2^2 , one obtains a new interval $R_{N, (p_1)} = [[3, 3^2], (n_{(p_1)} \leq N)]$ consisting of the natural numbers $n_{(p_1)} \geq 3$ (which are not divisible by 2) and the first prime $p_1 = 2$, which lies outside this interval $R_{N, (p_1)}$. All natural numbers $n_{(p_1)}$, which lie within the interval $[3, 3^2)$ are the primes p_2, p_3, p_4 in their correct order starting with p_2 .

In step $m \geq 2$ one adds to the result received from step $m - 1$ as interval $R_{N,(p_1,\dots,p_{m-1})} = \left[\left[p_m, p_m^2 \right), (n_{(p_1,\dots,p_{m-1})} \leq N) \right]$ from the natural numbers $n_{(p_1,\dots,p_{m-1})} \geq p_m$ (which are not divisible by p_1, \dots, p_{m-1}) with the primes p_1, \dots, p_{m-1} which lie outside the interval $R_{N,(p_1,\dots,p_{m-1})}$ all missing natural numbers n , after that $n_{(p_1)}$ (which are not divisible by p_1), $n_{(p_1,p_2)}$ (which are not divisible by p_1, p_2), ..., up to $n_{(p_1,\dots,p_{m-2})}$ (which are not divisible by p_1, \dots, p_{m-2}) in order to be able to subtract from the interval $R_{N,(p_1,\dots,p_{m-1})}$ all multiples of the prime number p_m starting from p_m^2 after that. Then one subtracts from this sum all multiples of the prime number p_m starting from p_m^2 and obtains a new interval $R_{N,(p_1,\dots,p_m)} = \left[\left[p_{m+1}, p_{m+1}^2 \right), (n_{(p_1,\dots,p_m)} \leq N) \right]$ consisting of the natural numbers $n_{(p_1,\dots,p_m)} \geq p_{m+1}$ (which are not divisible by p_1, \dots, p_m) with the primes p_1, \dots, p_m , which lie outside the interval $R_{N,(p_1,\dots,p_m)}$. The prime p_m is $\leq \sqrt{N}$. All natural numbers $n_{(p_1,\dots,p_m)}$ which lie within the interval $\left[p_{m+1}, p_{m+1}^2 \right)$ are all the primes in their correct order from p_{m+1} up to $p_{x_m} < p_{m+1}^2$. Here p_{x_m} is the last of the prime numbers.

All subsequent lines of the formula in the term of the addition are written in the same manner:

$$\left\{ \begin{array}{l} R_N = \left(\left\lfloor \frac{p_m}{2} \right\rfloor, \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \\ 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1)} = \left(\left\lfloor \frac{p_m}{3} \right\rfloor, \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \\ 3 \cdot \left(3, 5, 7, \dots, \left(n_{(p_1)} \leq \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1,p_2)} = \left(\left\lfloor \frac{p_m}{5} \right\rfloor, \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \\ 5 \cdot \left(5, 7, 11, \dots, \left(n_{(p_1,p_2)} \leq \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N,(p_1,p_2,p_3)} = \left(\left\lfloor \frac{p_m}{7} \right\rfloor, \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \\ 7 \cdot \left(7, 11, 13, \dots, \left(n_{(p_1,p_2,p_3)} \leq \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ \dots \\ \\ R_{N,(p_1,\dots,p_{m-2})} = \left(\left\lfloor \frac{p_m}{p_{m-1}} \right\rfloor, \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \\ p_{m-1} \cdot \left(p_{m-1}, p_m, p_{m+1}, \dots, \left(n_{(p_1,\dots,p_{m-2})} \leq \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \right) \cdot p_m \end{array} \right.$$

All intervals from above can be described as

$$R_N = \left(\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor, \left\lfloor \frac{N}{p_{\nu-1} \cdot p_m} \right\rfloor \right) \text{ for } \nu = 2; m = 2$$

and

$$R_{N,(p_1,\dots,p_{\nu-2})} = \left(\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor, \left\lfloor \frac{N}{p_{\nu-1} \cdot p_m} \right\rfloor \right) \text{ for } \nu = 3, 4, 5, \dots, m; m \geq 3.$$

The interval $R_{N,(p_1,\dots,p_{\nu-2})}$ has to be regarded as

$$\left\lfloor \frac{p_m}{p_{\nu-1}} \right\rfloor < n_{(p_1,\dots,p_{\nu-2})} \leq \left\lfloor \frac{N}{p_{\nu-1} \cdot p_m} \right\rfloor.$$

(The explanation for the interval R_N is analogous to the one for the interval $R_{N,(p_1,\dots,p_{\nu-2})}$. Thus it is not explained again here.)

That is:

- (1) if the value $\lfloor p_m/p_{\nu-1} \rfloor$ is not contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, the value of the number next in size is taken which is available in the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.
- (2) if the value $\lfloor p_m/p_{\nu-1} \rfloor$ is contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$ also the value of the number next in size is taken which is available in the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.
- (3) if the value $\lfloor N/(p_{\nu-1} \cdot p_m) \rfloor$ is contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, it remains as such.
- (4) if the value $\lfloor N/(p_{\nu-1} \cdot p_m) \rfloor$ is not contained within the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$, the value of the next smaller number is taken which is available in the interval $R_{N,(p_1,\dots,p_{\nu-2})}$ of the natural numbers $n_{(p_1,\dots,p_{\nu-2})}$.

Now all multiples of the prime p_m starting with p_m^2 , being the natural numbers n , can be subtracted from the interval $R_{N,(p_1,\dots,p_{m-1})}$. These natural numbers one receives from the product:

$$p_m \cdot \left(p_m, (p_m + 1), (p_m + 2), \dots, \left(n \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \right).$$

One has to consider Remark 1 on page 4 too.

3.2. The first derivation of the formula for the ‘‘Sieve of Eratosthenes’’ in its development from the first step to step m. First step $m = 1$.

$$\begin{aligned} & R_{N,(p_1)=2, \left[\left[3, 3^2 \right], (n_{(p_1)} \leq N) \right]} \\ & 2, (3, 5, 7, 9, \dots, (n_{(p_1)} \leq N)) = \\ & = R_{N=\left[\left[2, 2^2 \right], (n \leq N) \right]} - 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right) \right), \end{aligned}$$

step $m \geq 2$.

$$\begin{aligned}
& 2, 3, 5, \dots, p_m, R_{N, (p_1, \dots, p_m)} = 2, 3, 5, \dots, p_m, \left[[p_{m+1}, p_{m+1}^2], (n_{(p_1, \dots, p_m)} \leq N) \right] \\
& 2, 3, 5, \dots, p_m, (p_{m+1}, p_{m+2}, p_{m+3}, \dots, p_{x_m}, p_{m+1}^2, \dots, (n_{(p_1, \dots, p_m)} \leq N)) = \\
& 2, 3, 5, \dots, p_{m-1}, R_{N, (p_1, \dots, p_{m-1})} = 2, 3, 5, \dots, p_{m-1}, \left[[p_m, p_m^2], (n_{(p_1, \dots, p_{m-1})} \leq N) \right] \\
& = 2, 3, 5, \dots, p_{m-1}, (p_m, p_{m+1}, p_{m+2}, \dots, p_{x_{m-1}}, p_m^2, \dots, (n_{(p_1, \dots, p_{m-1})} \leq N)) + \\
& + \left\{ \begin{array}{l} R_N = \left(\left\lfloor \frac{p_m}{2} \right\rfloor, \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \\ 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N, (p_1)} = \left(\left\lfloor \frac{p_m}{3} \right\rfloor, \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \\ 3 \cdot \left(3, 5, 7, \dots, \left(n_{(p_1)} \leq \left\lfloor \frac{N}{3 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N, (p_1, p_2)} = \left(\left\lfloor \frac{p_m}{5} \right\rfloor, \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \\ 5 \cdot \left(5, 7, 11, \dots, \left(n_{(p_1, p_2)} \leq \left\lfloor \frac{N}{5 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_{N, (p_1, p_2, p_3)} = \left(\left\lfloor \frac{p_m}{7} \right\rfloor, \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \\ 7 \cdot \left(7, 11, 13, \dots, \left(n_{(p_1, p_2, p_3)} \leq \left\lfloor \frac{N}{7 \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ \dots \\ R_{N, (p_1, \dots, p_{m-2})} = \left(\left\lfloor \frac{p_m}{p_{m-1}} \right\rfloor, \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \\ p_{m-1} \cdot \left(p_{m-1}, p_m, p_{m+1}, \dots, \left(n_{(p_1, \dots, p_{m-2})} \leq \left\lfloor \frac{N}{p_{m-1} \cdot p_m} \right\rfloor \right) \right) \cdot p_m \\ \\ R_N = \left[p_m, \left\lfloor \frac{N}{p_m} \right\rfloor \right] \\ - p_m \cdot \left(p_m, (p_m + 1), (p_m + 2), \dots, \left(n \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \right). \end{array} \right. \quad - \\
\end{aligned} \tag{2}$$

The programme for Chapter 3 (“derivationI.py”) can be found in the Appendix, page 13.

4. THE SECOND DERIVATION OF THE FORMULA OF THE “SIEVE OF ERATOSTHENES” IN ITS DEVELOPMENT FROM THE FIRST STEP UP TO STEP M

4.1. Description of the procedure in the formula. In the first step $N \geq 2$, a given natural number, is chosen and the natural numbers n in the interval $R_N = [[2, 2^2], (n \leq N)]$ which do not exceed the value of the number N . If first subtracting from the interval R_N all even natural numbers $n \geq 2$ starting with 2^2 , one obtains a new interval $R_{N, (p_1)} = [[3, 3^2], (n_{(p_1)} \leq N)]$ consisting of the natural numbers $n_{(p_1)} \geq 3$ (which are not divisible by $p_1 = 2$) and the first prime $p_1 = 2$, which lies outside this interval $R_{N, (p_1)}$. All natural numbers $n_{(p_1)}$, which lie within the interval $[3, 3^2)$ are the primes p_2, p_3, p_4 in their correct order starting with p_2 .

In step $m \geq 2$ one subtracts from the result received from step $m - 1$ as interval $R_{N, (p_1, \dots, p_{m-1})} = [[p_m, p_m^2], (n_{(p_1, \dots, p_{m-1})} \leq N)]$ consisting of the natural numbers $n_{(p_1, \dots, p_{m-1})} \geq p_m$ (which are not divisible by p_1, \dots, p_{m-1}) with the primes

p_1, \dots, p_{m-1} which lie outside the interval $R_{N,(p_1, \dots, p_{m-1})}$ all the natural numbers $n_{(p_1, \dots, p_{m-1}) \geq p_m}$ divisible by the prime p_m starting with p_m^2 . One obtains a new interval $R_{N,(p_1, \dots, p_m)} = [[p_{m+1}, p_{m+1}^2], (n_{(p_1, \dots, p_m)} \leq N)]$ consisting of the natural numbers $n_{(p_1, \dots, p_m)} \geq p_{m+1}$ (which are not divisible by p_1, \dots, p_m) with the primes p_1, \dots, p_m , which lie outside the interval $R_{N,(p_1, \dots, p_m)}$. The prime p_m is $\leq \sqrt{N}$. All natural numbers $n_{(p_1, \dots, p_m)}$ which lie within the interval $[p_{m+1}, p_{m+1}^2]$ are all the primes in their correct order from p_{m+1} up to $p_{x_m} < p_{m+1}^2$. Here p_{x_m} is the last of the prime numbers.

One has to consider Remark 1 on page 4 too.

Note 1. *Derivation II is easier than the other two formulae in this paper since in its calculation there is only one subtraction in which one uses the same natural numbers $n_{(p_1, \dots, p_{m-1}) \geq p_m}$ in the minuend and subtrahend, but which have different upper bounds, namely N and $\lfloor N/p_m \rfloor$.*

4.2. The second derivation of the formula of the ‘‘Sieve of Eratosthenes’’ in its development from step one up to step m. First step $m = 1$.

$$\begin{aligned} & 2, R_{N,(p_1)=2}, [[3, 3^2], (n_{(p_1)} \leq N)] \\ & 2, (3, 5, 7, 9, \dots, (n_{(p_1)} \leq N)) = \\ & = \begin{matrix} R_N = [[2, 2^2], (n \leq N)] \\ 2, 3, 4, \dots, (n \leq N) \end{matrix} - 2 \cdot \begin{matrix} R_N = [[2, 2^2], (n \leq \lfloor \frac{N}{2} \rfloor)] \\ 2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right) \end{matrix} \end{aligned}$$

step $m \geq 2$.

$$\begin{aligned} & 2, 3, 5, \dots, p_m, R_{N,(p_1, \dots, p_m)} = 2, 3, 5, \dots, p_m, [[p_{m+1}, p_{m+1}^2], (n_{(p_1, \dots, p_m)} \leq N)] \\ & 2, 3, 5, \dots, p_m, (p_{m+1}, p_{m+2}, p_{m+3}, \dots, p_{x_m}, p_{m+1}^2, \dots, (n_{(p_1, \dots, p_m)} \leq N)) = \\ & = 2, 3, 5, \dots, p_{m-1}, R_{N,(p_1, \dots, p_{m-1})} = 2, 3, 5, \dots, p_{m-1}, [[p_m, p_m^2], (n_{(p_1, \dots, p_{m-1})} \leq N)] \\ & = 2, 3, 5, \dots, p_{m-1}, (p_m, p_{m+1}, p_{m+2}, \dots, p_{x_{m-1}}, p_m^2, \dots, (n_{(p_1, \dots, p_{m-1})} \leq N)) - \\ & - p_m \cdot \begin{matrix} R_{N,(p_1, \dots, p_{m-1})} = [[p_m, p_m^2], (n_{(p_1, \dots, p_{m-1})} \leq \lfloor \frac{N}{p_m} \rfloor)] \\ p_m, p_{m+1}, p_{m+2}, \dots, \left(n_{(p_1, \dots, p_{m-1})} \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \end{matrix}. \end{aligned} \tag{3}$$

The programme for this Chapter (‘‘derivationII.py’’) can be found in the Appendix, page 14.

5. COROLLARY

Let $N \geq 2$ be the given value of the last member of the numerical sequence of the numbers R_N of members consisting of the natural numbers $n \geq 2$, which lie within the interval $[[2, 2^2], (n \leq N)]$. If one has subtracted from the numerical sequence R_N the numerical sequence R_N of members being 2 times enlarged and consisting of natural numbers $n \geq 2$, which lie within the interval $[[2, 2^2], (n \leq \lfloor N/2 \rfloor)]$,

one subtracts after that the 3 times enlarged numerical sequence $R_{N,(p_1)}$ of members consisting of natural numbers $n_{(p_1)} \geq 3$, (which are not divisible by p_1) which lie within the interval $[[3, 3^2], (n_{(p_1)} \leq \lfloor N/3 \rfloor)]$. Then one subtracts the 5 times enlarged numerical sequence $R_{N,(p_1,p_2)}$ of members consisting of natural numbers $n_{(p_1,p_2)} \geq 5$ (which are not divisible by p_1, p_2), which lie within the interval $[[5, 5^2], (n_{(p_1,p_2)} \leq \lfloor N/5 \rfloor)]$, ..., and thus for the last prime $p_m \leq \sqrt{N}$ one subtracts the p_m times enlarged numerical sequence $R_{N,(p_1,\dots,p_{m-1})}$ of members consisting of natural numbers $n_{(p_1,\dots,p_{m-1})} \geq p_m$ (which are not divisible by p_1, \dots, p_{m-1}), which lie within the interval $[[p_m, p_m^2], (n_{(p_1,\dots,p_{m-1})} \leq \lfloor N/p_m \rfloor)]$. Having done this one obtains the numerical sequence $R_{N,(p_1,\dots,p_m)}$ of members consisting of natural numbers $n_{(p_1,\dots,p_m)} \geq p_{m+1}$ (which are not divisible by p_1, \dots, p_m), which lie within the interval $[[p_{m+1}, p_{m+1}^2], (n_{(p_1,\dots,p_m)} \leq N)]$ with the primes p_1, \dots, p_m , which lie outside the interval $R_{N,(p_1,\dots,p_m)}$.

The same numerical sequence $R_{N,(p_1,\dots,p_m)}$ of members consisting of natural numbers $n_{(p_1,\dots,p_m)} \geq p_{m+1}$ (which are not divisible by p_1, \dots, p_m), which lie within the interval $[[p_{m+1}, p_{m+1}^2], (n_{(p_1,\dots,p_m)} \leq N)]$ with the primes p_1, \dots, p_m , which lie outside the interval $R_{N,(p_1,\dots,p_m)}$ one obtains from the numerical sequence $R_{N,(p_1,\dots,p_{m-1})}$ of members consisting of natural numbers $n_{(p_1,\dots,p_{m-1})} \geq p_m$ (which are not divisible by p_1, \dots, p_{m-1}), which lie within the interval $[[p_m, p_m^2], (n_{(p_1,\dots,p_{m-1})} \leq N)]$ with the primes p_1, \dots, p_{m-1} , which lie outside the interval $R_{N,(p_1,\dots,p_{m-1})}$. All natural numbers $n_{(p_1,\dots,p_m)}$ which lie within the interval $[p_{m+1}, p_{m+1}^2]$ are primes in their correct order starting from p_{m+1} and ending at $p_{x_m} < p_{m+1}^2$. Here p_{x_m} is the last of the prime numbers.

Proof. One considers all intervals $R_N, R_{N,(p_1)}, R_{N,(p_1,p_2)}, \dots, R_{N,(p_1,\dots,p_m)}$ of the corresponding natural numbers $n, n_{(p_1)}, n_{(p_1,p_2)}, \dots, n_{(p_1,\dots,p_m)}$ as the numerical sequence with the corresponding members of the natural numbers $n, n_{(p_1)}, n_{(p_1,p_2)}, \dots, n_{(p_1,\dots,p_m)}$. Index p_1, \dots, p_m for the members of the numerical sequences means that these members are not divisible by the primes p_1, \dots, p_m , which is valid for all $m \geq 1$. One proves that the formula, written for a specified amount of steps $m \geq 1$, is the same as if it was written for a certain step $m \geq 1$.

Therefore:

Step $m = 1$.

$$\begin{aligned} & 2, R_{N,(p_1)=2}, [[3, 3^2], (n_{(p_1)} \leq N)] \\ & 2, (3, 5, 7, 9, \dots, (n_{(p_1)} \leq N)) = \\ & = \frac{R_N = [[2, 2^2], (n \leq N)]}{2, 3, 4, \dots, (n \leq N)} - 2 \cdot \left(\frac{R_N = [[2, 2^2], (n \leq \lfloor \frac{N}{2} \rfloor)]}{2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right)} \right), \end{aligned}$$

step $m \geq 2$.

$$\begin{aligned} & 2, 3, 5, \dots, p_m, R_{N,(p_1,\dots,p_m)=2,3,5,\dots,p_m}, [[p_{m+1}, p_{m+1}^2], (n_{(p_1,\dots,p_m)} \leq N)] \\ & 2, 3, 5, \dots, p_m, (p_{m+1}, p_{m+2}, p_{m+3}, \dots, p_{x_m}, p_{m+1}^2, \dots, (n_{(p_1,\dots,p_m)} \leq N)) = \\ & = \frac{R_N = [[2, 2^2], (n \leq N)]}{2, 3, 4, \dots, (n \leq N)} - \end{aligned}$$

$$\begin{aligned}
& - 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right) \right) - \\
& - 3 \cdot \left(3, 5, 7, \dots, \left(n_{(p_1)} \leq \left\lfloor \frac{N}{3} \right\rfloor \right) \right) - \\
& - 5 \cdot \left(5, 7, 11, \dots, \left(n_{(p_1, p_2)} \leq \left\lfloor \frac{N}{5} \right\rfloor \right) \right) - \dots \\
& - p_m \cdot \left(p_m, p_{m+1}, p_{m+2}, \dots, \left(n_{(p_1, \dots, p_{m-1})} \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \right) = \\
& = 2, 3, 5, \dots, p_{m-1}, \left(p_m, p_{m+1}, p_{m+2}, \dots, p_{x_{m-1}}, p_m^2, \dots, \left(n_{(p_1, \dots, p_{m-1})} \leq N \right) \right) - \\
& - p_m \cdot \left(p_m, p_{m+1}, p_{m+2}, \dots, \left(n_{(p_1, \dots, p_{m-1})} \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \right).
\end{aligned} \tag{4}$$

One has to consider Remark 1 on page 4 too. \square

That one obtains the correct results with this formula is proven by the programme which is written for this formula. All results for a given natural number $N=10,000,000$, which are obtained with the help of this programme, are absolutely correct.

The proofs for the formula and its first derivation are analogous.

Note 2. *The first part of the above Formula (4) one can use singly and step by step in order to find all prime numbers in their correct order one after the other. For all $m \geq 1$ step by step:*

$$\begin{aligned}
& 2, 3, 5, \dots, p_m, R_{N, (p_1, \dots, p_m)} = 2, 3, 5, \dots, p_m, \left[[p_{m+1}, p_{m+1}^2], \left(n_{(p_1, \dots, p_m)} \leq N \right) \right] \\
& 2, 3, 5, \dots, p_m, \left(p_{m+1}, p_{m+2}, p_{m+3}, \dots, p_{x_m}, p_{m+1}^2, \dots, \left(n_{(p_1, \dots, p_m)} \leq N \right) \right) = \\
& = R_{N, (p_1, \dots, p_m)} = [2, 2^2], (n \leq N) \\
& - 2, 3, 4, \dots, \left(n \leq N \right) - \\
& - 2 \cdot \left(2, 3, 4, \dots, \left(n \leq \left\lfloor \frac{N}{2} \right\rfloor \right) \right) - \\
& - 3 \cdot \left(3, 5, 7, \dots, \left(n_{(p_1)} \leq \left\lfloor \frac{N}{3} \right\rfloor \right) \right) - \\
& - 5 \cdot \left(5, 7, 11, \dots, \left(n_{(p_1, p_2)} \leq \left\lfloor \frac{N}{5} \right\rfloor \right) \right) - \dots \\
& - p_m \cdot \left(p_m, p_{m+1}, p_{m+2}, \dots, \left(n_{(p_1, \dots, p_{m-1})} \leq \left\lfloor \frac{N}{p_m} \right\rfloor \right) \right).
\end{aligned}$$

6. APPENDIX: SOURCE CODES FOR PROGRAMME 'PYTHON 2.7'

The following programmes are written in Python 2.7. The software can be downloaded from the internet (www.python.org). All programme files can be downloaded from www.maraev.de.

6.1. Source code for Chapter 2: The formula for the “Sieve of Eratosthenes” (“eratosthenes.py”).

```
#!/usr/bin/python

from NumSequence import *
import sys

if len ( sys.argv ) > 1:
    N = int ( sys.argv[1] )
else:
    N = 100

print "N:", N

R = NumSequence ( 2, N + 1 )

# current prime number
P = 2

Primes = []

I = 0

RN = R

Sieves = [ R ]
while True:
    R = R - P * RN[2 : N/P+1]

    if P * P >= N:
        break

    Primes.append ( R.pop_front ( ) )

    I = I + 1
    Sieves.append ( R )

    P = R[0]

    R = R + P * RN[ 2 : min (P, N/P+1) ]
```

```

    i = 0
    while i < I:
        R = R + Primes[i] * P * Sieves[i][ (P / Primes[i]) + 1 : \
            (N/(Primes[i]*P)) + 1 ]
        i = i + 1

Primes.extend ( R )

#print "Primes:", Primes

import primes_ref

primes_ref.verify ( Primes, N )

```

6.2. Source code for Chapter 3: The first derivation of the formula for the “Sieve of Eratosthenes” (“derivationI.py”).

```

#! /usr/bin/python

from NumSequence import *
import sys

if len ( sys.argv ) > 1:
    N = int ( sys.argv[1] )
else:
    N = 100

print "N:", N

R = NumSequence ( 2, N + 1 )

# current prime number
P = 2

Primes = []

I = 0

RN = R

Sieves = [ R ]

while P * P <= N:
    R = R - P * RN[P : N/P+1]

    Primes.append ( R.pop_front () )

    I = I + 1

```

```

Sieves.append ( R )

P = R[0]

i = 0
while i < I:
    R = R + Primes[i] * P * Sieves[i][ ( P / Primes[i] ) + 1 : \
        (N/(Primes[i]*P)) + 1 ]

    i = i + 1

Primes.extend ( R )

#print "Primes:", Primes

import primes_ref

primes_ref.verify ( Primes, N )

```

6.3. Source code for Chapter 4: The second derivation of the formula for the “Sieve of Eratosthenes” in its development from step one up to step m (“derivationII.py”).

```

#!/usr/bin/python

from NumSequence import *
import sys

if len ( sys.argv ) > 1:
    N = int ( sys.argv[1] )
else:
    N = 100

print "N:", N

R = NumSequence ( 2, N + 1 )

# current prime number
P = 2

Primes = []

while P * P < N:
    R = R - P * R[2 : N/P+1]

    Primes.append ( R.pop_front ( ) )

    P = R[0]

```

```

Primes.extend ( R )

#print "Primes:", Primes

import primes_ref

primes_ref.verify ( Primes, N )

```

6.4. Source code to define the mathematics in this paper, to be used with each of the programmes (“NumSequence.py”).

```

#! /usr/bin/python

import sys

class NumSequence:

    def __init__ ( self, *arg ):
        if len ( arg ) == 1:
            arg = arg[0]
            if isinstance ( arg, NumSequence ):
                self.S = arg.S
            elif type ( arg ) is list:
                self.S = arg
        elif len ( arg ) == 2:
            self.S = range ( arg[0], arg[1] )
        else:
            print "NumSequence: Parameter error"
            sys.exit ( 1 )

    def __repr__ ( self ):
        return repr ( self.S )

    def __add__ ( self, rhs ):
        if len ( self.S ) == 0:
            return NumSequence ( rhs )
        if len ( rhs.S ) == 0:
            return NumSequence ( self )

        S_iter = iter ( self.S ) # Left hand side iterator
        T_iter = iter ( rhs.S ) # Right hand side iterator
        R = list ( ) # Result

        s = S_iter.next ( )
        t = T_iter.next ( )

        while True:
            if s < t:
                R.append ( s )
                try:

```

```

        s = S_iter.next ()
    except StopIteration:
        # Left hand side is empty. Copy remaining \
        # elements of rhs.
        R.append ( t )
        try:
            while True:
                R.append ( T_iter.next () )
        except StopIteration:
            pass
        return NumSequence ( R )
elif s == t:
    # Adding existing element.
    print "Addition: adding existing element \
        (" + str ( s ) + ")"
    sys.exit ( 1 )
else:
    R.append ( t )
    try:
        t = T_iter.next ()
    except StopIteration:
        # Right hand side is empty. Copy remain
        R.append ( s )
        try:
            while True:
                R.append ( S_iter.next () )
        except StopIteration:
            pass
        return NumSequence ( R )

def __sub__ ( self, rhs ):
    S_iter = iter ( self.S ) # Minuend iterator
    T_iter = iter ( rhs.S ) # Subtrahend iterator
    R = list () # Result

    try:
        t = T_iter.next ()
    except StopIteration:
        # Excluding empty sequence.
        print "Subtraction: RHS is empty sequence"
        x=y
        sys.exit ( 1 )

    while True:
        try:
            s = S_iter.next ()
        except StopIteration:

```

```

        # Source sequence is empty but T_iter is still valid.
        print "Subtraction: Excluding non-existing \
              value (" + str ( t ) + ")"
        sys.exit ( 1 )

    if s < t:
        R.append ( s )
    elif s == t:
        try:
            t = T_iter.next ( )
        except StopIteration:
            # Nothing more to exclude, copy remaining elements.
            try:
                while True:
                    R.append ( S_iter.next ( ) )
            except StopIteration:
                pass
            return NumSequence ( R )
    else:
        print "s > t!", s, t
        sys.exit ( 1 )

##
# Multiply each member of the sequence with @p scalar
#
def __rmul__ ( self, scalar ):
    I = iter ( self.S )
    R = list ( )
    try:
        while True:
            R.append ( scalar * I.next ( ) )
    except StopIteration:
        pass
    return NumSequence ( R )

##
# Create subsequence
def __getitem__ ( self, key ):
    if type ( key ) is slice:
        R = list ( )
        if key.step is not None:
            print "Slice stepping != 1 is not supported"
            sys.exit ( 1 )
        N = iter ( self.S )
        n = N.next ( )
        # Skip leading subsequence

```

```

while True:
    if n >= key.start: break
    n = N.next ()

# Copy subsequence
try:
    while True:
        if n >= key.stop: break
        R.append ( n )
        n = N.next ()
except StopIteration:
    pass
return NumSequence ( R )
elif type ( key ) is int:
    return self.S[key]
else:
    print "getitem: Invalid index"

def pop_front ( self ):
    n = self.S[0]
    self.S = self.S[1:]
    return n

def __len__ ( self ):
    return len ( self.S )

```

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